

Using Binomial Decision Trees to Solve Real-Option Valuation Problems

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Traditional decision analysis methods can provide an intuitive approach to valuing projects with managerial flexibility or real options. The discrete-time approach to real-option valuation has typically been implemented in the finance literature using a binomial lattice framework. Instead, we use a binomial decision tree with risk-neutral probabilities to approximate the uncertainty associated with the changes in the value of a project over time. Both methods are based on the same principles, but we use dynamic programming to solve the binomial decision tree, thereby providing a computationally intensive but simpler and more intuitive solution. This approach also provides greater flexibility in the modeling of problems, including the ability to include multiple underlying uncertainties and concurrent options with complex payoff characteristics.

Key words: decision analysis; real options; decision trees; binary approximations

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1. Introduction

Discounted cash flow (DCF) methods are commonly used for the valuation of projects and for decision making regarding investments in real assets. One of the most important limitations of DCF is that it fails to account for the value of managerial flexibility inherent in many types of projects. The options derived from managerial flexibility are commonly called “real options” to reflect their association with real assets rather than with financial assets. Although appealing from a theoretical perspective, the practical use of real-option valuation techniques in industry has been limited by the mathematical complexity of these techniques and the resulting lack of intuition associated with the solution process, or the restrictive assumptions required to obtain analytical solutions.

In this article we outline how traditional decision analysis tools can be used as an alternative to solve real-option valuation problems based on the ideas suggested by Copeland and Antikarov (2001) and further illustrated in Copeland and Tufano (2004). We do this by using a binomial decision tree to determine

the cash flows and probabilities that give the correct project values when discounted to each period and to each uncertain state. Project flexibilities, or real options, can then be modeled easily as decisions that affect these cash flows. This specification of project uncertainties, cash flows, and decisions allows the problem to be modeled and solved using commercially available decision tree software familiar to the decision analysis community. Our discussion expands on the ideas presented originally by Brandão and Dyer (2005) and illustrates the approach with several examples.

While many of these ideas are relatively straightforward and build on concepts suggested by Nau and McCardle (1991) and Smith and Nau (1995), we hope to make this material more accessible to decision analysts and to encourage additional work on the relationship between decision analysis and finance. Triantis and Borison (2001) provide an assessment of the use of options-based project valuation methods in practice and conclude that a modest evolution is occurring within some companies to support their

adoption. In particular, Triantis and Borison anticipate increasing convergence among the various real-option approaches, particularly the decision-analytic and option-pricing approaches. In that spirit we also review some basic option-pricing concepts that will be familiar to many readers but that are nonetheless included as a useful reference in the context of this discussion. We will also take care to discuss the underlying assumptions and limitations of these methods and to suggest when they might be a valuable addition to the decision-analysis tool kit when used appropriately.

The remainder of the article is organized as follows. Section 2 reviews the traditional approaches to project valuation. Section 3 outlines a decision tree approach to the real-option problem discussed by Copeland and Tufano (2004). Section 4 provides an extension of this approach to problems in which project cash flows over time are explicitly modeled and used as the basis for valuing real options. This approach is illustrated in §5 with a numerical example. In §6 we conclude with a discussion of the limitations of this approach and identify some areas for further research.

2. Background on Project Valuation

With the DCF approach to valuation, the net present value of a project is calculated by discounting the future expected cash flows at a discount rate that takes into account the risk of the project. In practice, this discount rate is often the weighted average cost of capital (WACC) for the firm, based on the assumption that both the firm and the project share identical market risks. While this assumption may be valid for projects that mimic the risks associated with the firm as a whole, it may not be appropriate for unusual or innovative investment projects. In such cases, the practitioner must exercise judgment in choosing an appropriate discount rate for the project. For a discussion of the issues associated with the selection of a project discount rate and the calculation of the WACC, see Grinblatt and Titman (1998, Chapters 10 and 12).

A major criticism of DCF is the implicit assumption that the project's outcome will be unaffected by future decisions of the firm, thereby ignoring any value that comes from managerial flexibility. Management flexibility is the ability to make decisions during the execution of a project so that expected returns

are maximized or expected losses are minimized. Examples of project flexibilities include expanding operations in response to positive market conditions, abandoning a project that is underperforming, deferring investment for a period of time, suspending operations temporarily, switching inputs or outputs, reducing the project scale, or resuming operations after a temporary shutdown. The incremental value of these options can only be determined using an option-pricing or decision analysis approach.

Option-pricing methods were first developed to value financial options. However, the potential application of these methods to the valuation of options on real assets was quickly identified, and hundreds of scholarly papers have been written on this topic. Nevertheless, applications of real-option valuation methods to practical problems have been limited by the mathematical complexity of the approach, by the restrictive theoretical assumptions required, and by their lack of intuitive appeal.

The mathematical complexity associated with real-option theory stems from the fact that the general problem requires a probabilistic solution to a firm's optimal investment decision policy, not only at present but also at all instances in time up to the maturity of its options. To solve this problem of dynamic optimization, the evolution of uncertainty in the value of the real asset over time is first modeled as a stochastic process. Then the value of the firm's optimal policy is a partial differential equation that is obtained as the solution to a value function represented by Bellman's principle of optimality, where appropriate boundary conditions reflect the initial conditions and terminal payoff characteristics. When closed-form mathematical solutions are unavailable, which is usually the case for more complex problems where the project may be subject to several sources of uncertainty and more than one type of option, numerical methods and discrete dynamic programming must be used to obtain a solution.

A discrete approximation to the underlying stochastic process can be developed to provide a transparent and computationally efficient model of the valuation problem. The first example of this approach is a binomial lattice model that converges weakly to a lognormal diffusion of stock prices, developed by Cox et al. (1979). A binomial lattice may be viewed as

a probability tree with binary chance branches, with the unique feature that the outcome resulting from moving up (u) and then down (d) in value is the same as the outcome from moving down and then up. Thus, this probability tree is recombining, since there are numerous paths to the same outcomes, which significantly reduces the number of nodes in the lattice. A binomial lattice and the corresponding binomial tree are shown in Figure 1, where S is the current market price of the asset and q is the probability of an upward move to Su .

The binomial lattice model can be used to accurately approximate solutions from the Black-Scholes-Merton continuous-time valuation model for financial options, with the added advantage of allowing a solution for the value of early-exercise American options, whereas the Black-Scholes-Merton model can value only European options.

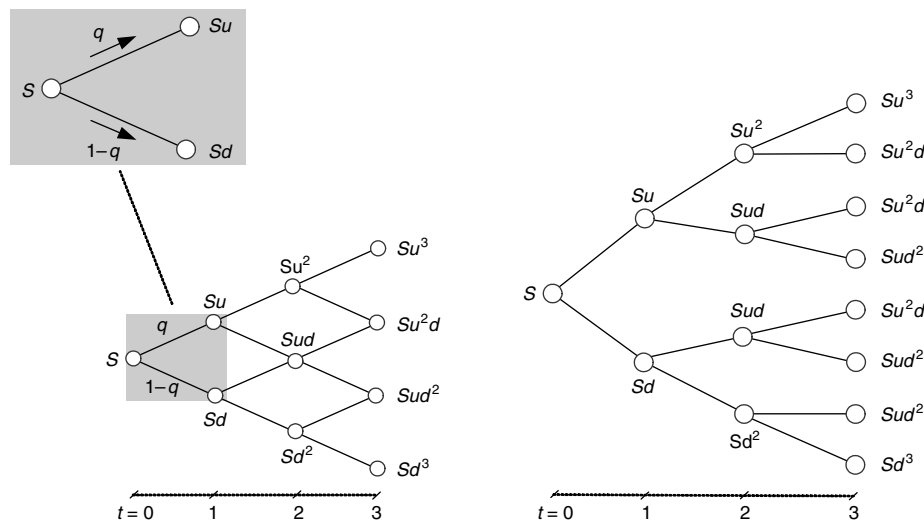
Unfortunately, the process of working through lattices can be cumbersome and nonintuitive, especially for more complex applications to real assets, which can involve several simultaneous and compound options. The typical approach to using a lattice involves finding a replicating portfolio at each node. This approach is based on traditional option-pricing methods, which require that markets be complete in the sense that there are enough traded assets to allow the creation of a portfolio of securities whose payoffs replicate the payoffs of the asset in all states

of nature and in all future periods. The assumption of the existence of a replicating portfolio underlies much of the initial work done in the field of continuous-time, real-option valuation by Brennan and Schwartz (1985), McDonald and Siegel (1986), Dixit and Pindyck (1994), and Trigeorgis (1996).

The use of traditional option-pricing methods and the replicating portfolio approach is complicated by the fact that, for most projects involving real assets, no such replicating portfolio of securities exists, so markets are incomplete. In this case, Dixit and Pindyck (1994) propose the use of dynamic programming using a subjectively defined discount rate, but the result does not provide a market value for the project and its options.

The application of decision analysis to real-option valuation problems seems natural because decision trees are commonly used to model project flexibility, but there has been limited work in this area (Howard 1996). Nau and McCardle (1991) and Smith and Nau (1995) study the relationship between option pricing theory and decision analysis and demonstrate that the two approaches yield the same results when applied correctly. Smith and Nau propose a method that integrates the two approaches by distinguishing between market risks, which can be hedged by trading securities and valued using option pricing theory, and private uncertainties, which are project-specific risks and can be valued using decision analysis techniques.

Figure 1 Recombining Binomial Lattice and Corresponding Binomial Tree



Smith and McCardle (1998, 1999) illustrate how this approach can be applied in the context of oil and gas projects and provide a discussion of lessons learned from applications to some case studies.

The distinction between market risks and project-specific risks is often a very natural one in oil and gas exploration projects, since oil and gas prices are market risks, while the project-specific risks may be the probability of a dry hole or the probability distribution regarding the volume of reserves. The McCardle-Nau-Smith approach (henceforth MNS) has a natural appeal in problem contexts such as these. However, there are projects in other industries where the distinction between market risks and project-specific risks is not as sharp.

Copeland and Antikarov (2001) have proposed a more general approach (henceforth CA) to valuing real options that may be applied to problems in cases where there is no market-traded asset. To obtain this generality, they make the assumption that the present value of the project without options is the best unbiased estimator of the market value of the project (the marketed asset disclaimer, or MAD assumption). Under this assumption, the value of the project without options serves as the underlying asset in the replicating portfolio, which implies that the markets are complete for the project with options. If the changes in the value of the project without options are then assumed to vary over time according to a random walk stochastic process, more formally called geometric Brownian motion (GBM), then the options can be valued with traditional option pricing methods.

These assumptions are conceptually similar to those adopted earlier by Luehrman (1998a, b) to rationalize the direct application of the classic Black and Scholes (1973) option-pricing model to real options. While Luehrman's approach has generally been discounted as too simplistic (Triantis and Borison 2001), the development by CA is more robust. For example, it allows for the modeling of project cash flows and other project-specific risks to capture a more realistic representation of the underlying problem, for the use of stochastic processes other than the GBM, and for the separation of market and private risks.

Copeland and Tufano (2004) have recently championed this approach in an article in *Harvard Business Review*, guaranteeing it high visibility among

practitioners. However, their presentation is based on the use of binomial lattices and the construction of market portfolios that replicate the risk characteristics of the project, and therefore it suffers from a lack of intuitive appeal. Adapting this method to use binomial decision trees provides transparency to its logic and offers a link to decision analysis approaches to real-options valuation.

3. A Decision Analysis Approach to Valuation

Decision tree analysis (DTA) can be used to model managerial flexibility in discrete time by constructing a tree with decision nodes that represent decisions the manager can make to maximize the value of the project as uncertainties are resolved over the project's life. This approach allows some of the limitations of the static DCF approach to be overcome. In fact, a naïve approach to valuing projects with real options would be to simply include decision nodes corresponding to project options into a decision tree model of the project uncertainties and solve the problem using the same risk-adjusted discount rate judged to be appropriate for the original project without options.

However, the naïve approach does not provide a correct valuation of the real options. This is because the optimization that occurs at the decision nodes changes the expected future cash flows, thereby altering the risk characteristics of the project. Thus, the standard deviation of the project cash flows with flexibility is different from that of the project without flexibility, and the risk-adjusted discount rate for the project without options may not be appropriate after the real options have been included in the model. This observation has caused some authors to incorrectly conclude that DTA cannot be used to value real options (e.g., see the discussion in CA 2001). However, as noted by Smith (1999), the differences between the DTA and finance approaches are largely matters of style, and DTA can readily be augmented to incorporate market information about risk.

To adjust the naïve approach, we can use the replicating portfolio method to determine the correct discount rates for the project and thereby capture the market information about project risks. Let us first

assume there is a project of an *unknown* value V and a replicating portfolio of an amount A of a market-traded stock with a current price S and of B dollars invested in a risk-free bond that pays an interest rate r . For simplicity, we assume that for a one-period model with probability q the stock price will move up to Su at the end of the period, and with probability $1 - q$ it will move down to Sd , where u is a number greater than 1 reflecting a proportional increase in the stock value, and $d = 1/u$ is a number smaller than 1 reflecting a proportional decrease. This approach can be extended to multiple time periods by simply continuing to apply these same percentage changes to the values determined at the end of the one-period model, as we will show later.

The value of this portfolio one period from now will be $ASu + B(1 + r)$ and $ASd + B(1 + r)$ in the up and down states, respectively, and we assume that the values of the project in these same up and down states, V_u and V_d , are known. The dynamics of the stock, the bond, and the replicating portfolio are shown in Figure 2. For these portfolio values to replicate the value of the project in each of the up and down states exactly, the appropriate values of A and B must be determined by solving a system of two equations in two unknowns, $V_u = ASu + (1 + r)B$ and $V_d = ASd + (1 + r)B$, which yield $A = (V_u - V_d)/((u - d)S)$ and $B = (uV_d - dV_u)/((u - d)(1 + r))$. If the holdings A and B are the replicating portfolio for the project at the end of the period, then by the basic no-arbitrage argument of finance theory, their current price, $AS + B$, must also be the price, or value, of the project V . While this form of the replicating portfolio method provides a market-based adjustment for the risk in the project, for a multiperiod and multistate project this proves to be cumbersome computationally, since this exercise must be repeated for each node of the lattice.

Note that the expressions for A and B do not include the probability q of an up move in the stock price, which eliminates the necessity of trying to estimate this variable. This is an important advantage of this approach to valuation, since it relies only on information that can be calculated from market data. If the value of q were known, then the appropriate discount rate for the project could be found by solving the relationship between the expected future value and the current value V of the project for k , as shown in Equation (1).

$$V = \frac{quV_u + (1 - q)dV_d}{1 + k} \tag{1}$$

Fortunately, there is an equivalent but simpler pricing algorithm that is analogous to the replicating portfolio approach and that avoids the need to estimate q or k .

In this alternative approach, we account for the project risk by adjusting the up and down probabilities rather than by adjusting the discount rate. The discount rate in Equation (1) is set equal to the risk-free rate of interest r , which is known, and (1) is solved for the value of the implied probability p instead of the value k . Since the risk-free rate r will be less than the risk-adjusted discount rate k , the derived probability p will be less than the true probability of the “up state,” q .

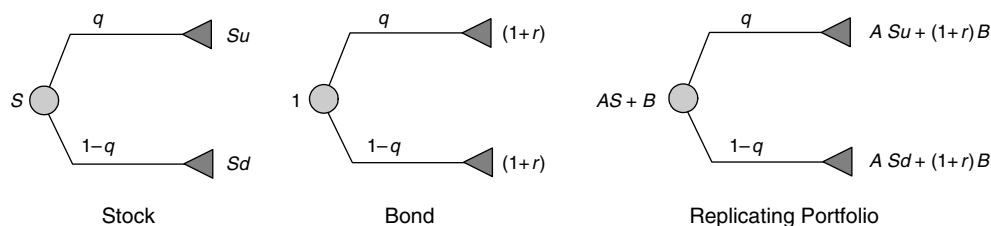
The solution for p is easily obtained by using the relationship $V = AS + B$ and substituting the values for A and B determined above. The resulting equation for the current value of the project is:

$$V = \left[\left(\frac{1 + r - d}{u - d} \right) V_u + \left(\frac{u - (1 + r)}{u - d} \right) V_d \right] / (1 + r) \quad \text{or}$$

$$V = \frac{[pV_u + (1 - p)V_d]}{1 + r},$$

where $p = (1 + r - d)/(u - d)$. These values are often called “risk-neutral” probabilities because assets are

Figure 2 Dynamics of the Stock Price, the Bond Yield, and the Project Value



priced as if there is a risk-neutral representative investor with an estimate of probability p for the up state.

This risk-neutral probability p can then be used in a binomial lattice or tree to calculate an expected value given the future payoffs, and the risk-free rate can be used to discount the future payoffs. This shifts the problem of finding the appropriate risk-adjusted discount rate for a project to the problem of finding the appropriate risk-neutral probabilities to use in calculating the risk-neutral measure of value. Fortunately, the latter problem is often easier to solve, since these risk-neutral probabilities may be available from market data or from assumptions based on theoretical arguments regarding the underlying stochastic process associated with the value of the project. Moreover, the GBM with constant volatility is the most common assumption regarding the stochastic process associated with the project value, and this implies that the values of p and $(1 - p)$ are constant and applied throughout the lattice or tree, whereas the values of A and B must be calculated for every node.

We demonstrate this approach by solving an example from a recent article by Copeland and Tufano (2004). In this example, a firm is considering a phased investment in a plant. An initial investment of \$60 million to cover the cost of permits and preparation for the effort is due immediately. At the end of one year, a commitment of \$400 million is required for the design phase of the new plant. Once the design is completed one year later, the firm would have a two-year window during which to make the final investment in the plant of \$800 million, which would pay for construction. If the firm decides not to invest during these two years, it then foregoes the opportunity to build the plant.

From the real-options perspective, this investment opportunity is a compound option. The initial payment of \$60 million gives the firm the option to continue with the project for one year, at the end of which it has the option to invest an additional \$400 million in the design phase. In turn, the completion of the design phase gives the firm the option to construct the plant at the end of year two or at the end of year three.

The firm estimates that if the plant existed today it would be worth \$1,000 million based on a traditional

net present value (NPV) calculation with the information currently available, but the value of this plant in future years is uncertain and is expected to change over time. The investments at the end of years one, two, and three are options and will be made only if they are justified by the revised estimate of the project at that point in time.

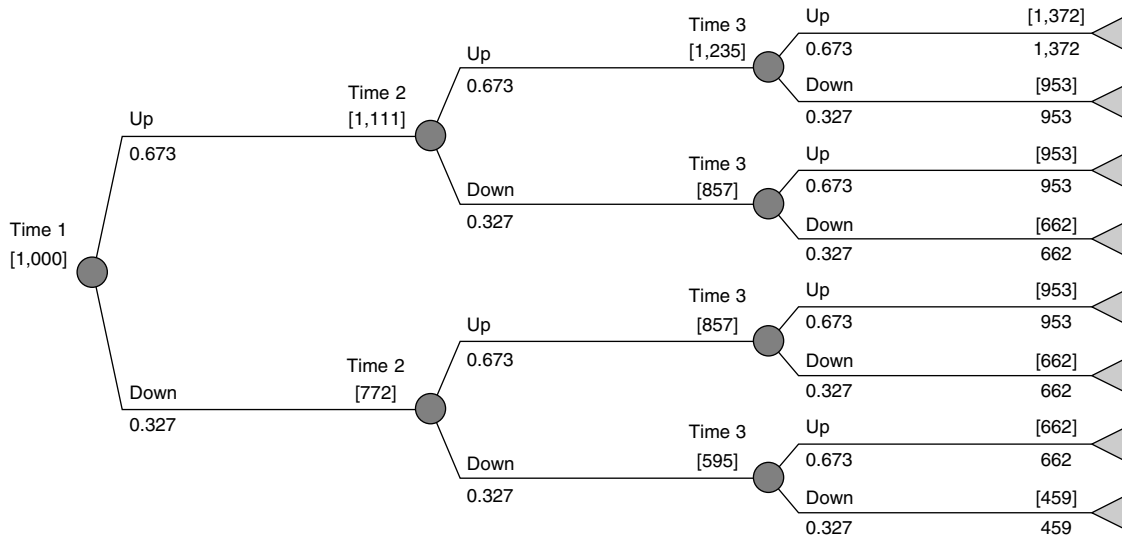
To carry out an analysis of this problem some assumptions must be made regarding the uncertainty in the future value of the project. A common assumption regarding stock prices is that current prices already incorporate all relevant information available at this point in time, and that future changes will be the effect of random and thus unpredictable shocks, which are modeled as a random walk. This assumption and other arguments support the use of a GBM to model the dynamic uncertainty associated with stock prices (Hull 2003). CA (2001, Chapter 8) use similar arguments to justify the use of the GBM to model changes in the value of a project over time in some instances, and it is used in this example for simplicity.

However, the assumption of the GBM model may not be appropriate in all situations, and it is not a requirement of the CA approach. In the discussion section we describe how alternate models of stochastic processes may be approximated using binomial lattices or trees and used with this approach.

The critical parameters required to model the GBM are the starting value, \$1,000 million in this example, the risk-free interest rate r , assumed to be 8% per year, and the volatility, denoted as σ , which is the annualized percentage standard deviation of the returns and is given as 18.23% in this example. This allows the computation of the values of p , u , and d , respectively the risk-neutral probabilities and the up and down proportional changes in the value per period illustrated in the previous example. With these parameters, this continuous-time stochastic process can be approximated with a discrete time binomial lattice. Copeland and Tufano assume this process represents the evolution of the project value, without options, over time and that this serves as the underlying asset (MAD assumption).

The idea behind the calculation of the parameters used in the binomial approximation of the stochastic process is relatively simple. If the value of the project

Figure 3 The Binomial Tree Approximation to the Geometric Brownian Motion



is assumed to follow a GBM, then the estimate of its value at any point in time has a lognormal distribution. By equating the first and second moments of a binomial and a lognormal distribution, we can calculate the corresponding values of u and d , and thus $V_u = Vu$ and $V_d = Vd$, for each branch of the binomial approximation to ensure that the discrete distribution approximates its continuous counterpart in the limit as Δt becomes small. Adding the convenient specification that $u = 1/d$ to the equations for matching the mean and variance of the GBM yields $u = e^{\sigma\sqrt{\Delta t}}$. We then obtain the risk-neutral probability $p = (1 + r\Delta t - d)/(u - d)$. In this example, we model three periods and choose $\Delta t = 1$. Therefore, $u = e^{0.1823} = 1.2$, $d = 0.83$, and $p = 0.673$. We emphasize again that only three parameters are needed to specify this discrete approximation to the GBM estimate of the evolution of the uncertain project value over time: the estimate of the current value of this project, the volatility of the returns from the project, and the risk-free rate. For details associated with this binomial approximation, see Cox et al. (1979) or Hull (2003).

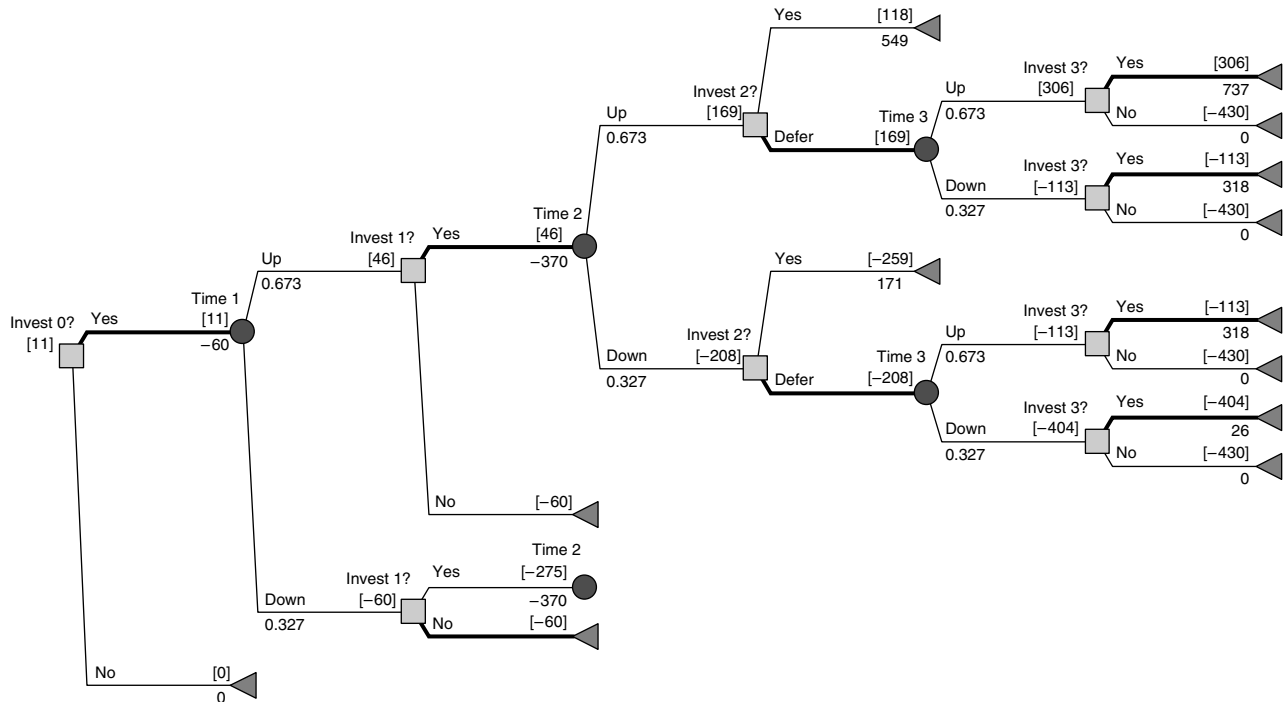
Copeland and Tufano (2004) solve this problem using a recombining binomial lattice and obtain the value of the options by calculating a replicating portfolio with values for A and B at each node in the lattice. The value of the project at any point in this lattice is given by $V_{i,j} = V_0 u^{i-j} d^j$. While this approach

is technically correct (given their assumptions), it is neither intuitively appealing nor computationally transparent.

The same parameters can be used in a decision tree with binary chance nodes to yield an equivalent binomial tree for the project value, as shown in Figure 3. The values shown at each node in the tree are discounted Year 3 values, instead of the actual values at each point. However, it can easily be verified that this binomial tree corresponds to the lattice developed by Copeland and Tufano. Notice, for example, that the value at the end of the up move in Time 1 and the down move in Time 2 is exactly equal to the value at the end of the down move in Time 1 and the up move in Time 2. These two nodes would be combined into one node in the corresponding binomial lattice.

The advantage of using the corresponding binomial tree rather than a binomial lattice can now be illustrated. The real options in the project can simply be modeled with decision nodes in the tree. This results in the tree in Figure 4, which shows that the expected value of the project with options is \$11 million after subtracting the initial investment cost. Notice, however, that the effort should be abandoned if the expected value of the project is lower at the end of the first time period, one year in this case. The approximation to the GBM could be improved by

Figure 4 Solution to the Real-Options Problem Using a Binomial Tree



adding additional periods of shorter duration at the expense of some computational burden.

This alternative approach yields the same optimal exercise policy and the project value of \$71 million shown in Copeland and Tufano (2004) prior to subtracting the investment cost. However, by using risk-neutral probabilities in a decision tree, we did not need to solve for the replicating portfolio at each node. Further, the optimal policy is obvious from the graphic view of the decision tree, whereas it must be inferred from a binomial lattice representation.

The decision analyst might remain somewhat skeptical at this point, however, since this approximation to the value of the project over time is based on the GBM assumption, and the volatility of 18.23% was simply given as one of the parameters for this problem. How might the volatility be derived in practice? One might conjecture that the source of this volatility would be associated with uncertainties in some underlying factors, such as sales volumes, prices, costs, and competitors' actions. Further, this analysis is focused on the change in the estimated value of the project and is very similar conceptually to the anal-

ysis of the value of a stock option using the Black and Scholes model. Therefore, it is similar in spirit to the simplistic option valuation approach suggested by Leuhrman (1998a). There are no allowances for changes in the cash flows over time, for the fact that the value of any project with a finite life will change as it is being executed or for options that occur during the operating life of the project.

A traditional decision tree analysis of this same problem might include estimates of the uncertainties associated with these underlying factors (sales volumes, prices, etc.) in the calculation of the present values for the project, highlighting what Smith (1999) has called an emphasis on modeling the sources of uncertainty in decision analysis versus an emphasis on modeling the dynamics of the uncertainty in real options. As we shall see, however, these same sources of uncertainty can be used to estimate the volatility of the project returns, and their impacts on cash flows over time can be modeled as well within this same generalized framework. The latter, in turn, allows the representation of real options that may occur during the operating life of the project.

4. Solving Real-Options Problems with Binomial Decision Trees

Building on the work of Nau and McCardle (1991), Smith and Nau (1995) suggested an approach for the valuation of real options using decision analysis techniques that differ in some significant ways from the one described above. The valuation procedure utilizes a separation of the project cash flows into two components, one subject to market risks and the other subject to private risks. Market risks depend only on market states and can be hedged by creating a replicating portfolio of traded securities. Private risks are project specific and thus cannot be hedged by trading securities. The market component is then valued using market information (risk-neutral probabilities), while the private component is valued using subjective beliefs and preferences (subjective probabilities and certainty equivalents). In this approach, as long as all the market risks can be hedged with a marketed commodity or security, there is no need to estimate a risk-adjusted discount rate for the project risks.

This approach is generalized in an integrated roll-back method. The steps of the procedure are as follows: (1) Calculate the NPV for all endpoints; (2) for chance nodes with private uncertainties, use the firm's subjective probabilities and exponential utility function; and (3) for chance nodes with market uncertainties, use risk-neutral probabilities inferred from market information. Smith and Nau (1995) demonstrate this approach for the example of a plant investment with two underlying uncertainties: future demand (which is correlated to a marketed security) and plant efficiency (private risk).

In many projects, some uncertainties fall somewhere in between the notions of private and market risks. For example, a pharmaceutical company's new drug development project may not include risks that can be replicated by a traded asset, but the price of the product is clearly a "market risk." Moreover, a project may have numerous uncertainties to model. Even if we can separate them into these two classes and establish replication for each individual market uncertainty, the underlying decision tree is computationally unwieldy since we must include a separate chance node for each uncertainty in each time period. Smith and McCardle (1999) refer to the latter

as a "dream tree" that cannot be solved because of its large size and suggest ways of trimming it.

An alternative to the construction of large trees with multiple uncertainties in each time period is the application of binomial decision trees to the approach proposed by CA (2001), illustrated in the previous example. In the discussion that follows, we will let V_i and C_i be random variables representing the uncertain project values and cash flows in period i , and \bar{V}_i and \bar{C}_i will be their corresponding means. Realizations of these random variables will be denoted with lower case, and the values associated with the discrete approximations to these random variables will be denoted as V_{ij} and C_{ij} , where j indicates an outcome state.

In this development, we make the assumption that the value of the project will evolve following a GBM process but describe alternative assumptions in the discussion. To show how this GBM assumption is utilized, let V_i be the value of a project at time period i and V_{i+1}/V_i be its return over the time period between i and $i+1$. Under the random walk assumption, the logarithm of the random return $z = \ln(V_{i+1}/V_i)$ is normally distributed, and we define \bar{z} and σ^2 as the mean and variance of this normal distribution. The assumption that the distribution of the logarithm of the project returns at any time is normal implies that the distribution of the project value at any time is lognormal. Therefore, V_i will be lognormally distributed and can be modeled as a GBM stochastic process in the form $dV = \mu V dt + \sigma V dw$, where $\mu = \bar{z} + (1/2)\sigma^2$ and $dw = \varepsilon\sqrt{dt}$ is a standard Wiener process. For a discussion of the random walk assumption, see also Hull (2003) and Luenberger (1998).

The assumption that project returns follow a random walk is important for projects that involve several uncertainties because it allows any number of uncertainties in the project model to be combined into a single representative uncertainty: the uncertainty associated with the stochastic process of the project value V . The parameters of this process can be obtained from a Monte Carlo simulation of the project cash flows. With these parameters, a discrete-time model using a binomial lattice or tree can be used to approximate the composite continuous-time stochastic process as before.

Consider a project that will last n periods that requires an initial investment I to be implemented and that generates an expected cash flow \bar{C}_i , $i = 1, 2, \dots, n$ in each of these periods. For simplicity we assume that the cash flows are paid instantaneously at the end of each time period in a manner analogous to stock dividends.

The problem is modeled in three steps. First, the expected present value of the project at Time 0 is calculated. Next, a Monte Carlo simulation is performed to combine several sources of uncertainty into a single representative uncertainty, which defines the stochastic process for the value of the project. The third and last step is to construct a binomial tree to model the dynamics of the project value using the parameters of the stochastic process and to add the decision nodes to model the project's real options.

These first two steps are identical to those proposed by CA (2001). For the third step we provide an alternative solution methodology based on a binomial tree that offers computational advantages and a more intuitive logic. For completeness, we briefly summarize the first two steps below and then discuss our proposed modifications of the third step in more detail.

Step 1

The expected present value of the project at Time 0, \bar{V}_0 , is determined using the traditional DCF method and without considering any managerial flexibility. This requires the estimation of the appropriate risk-adjusted discount rate for the project without options and introduces an element of judgment into this valuation approach (which we shall discuss subsequently). These cash flows are then discounted at this estimated risk-adjusted discount rate μ to obtain the expected present value of the project in each period:

$$\bar{V}_t = \sum_{i=t}^n \frac{\bar{C}_i}{(1 + \mu)^{i-t}}. \quad (2)$$

The expected present value of the project will decrease in each period as t increases if the cash flows are all positive, due to the payout of the cash flows in each period. Thus, for a project with finite life, the final value of the project will be 0.

The lognormal distribution of the project's value can be defined by the mean and standard deviation of

its returns. Under the MAD assumption, the present value of the project without options is taken as its market price, as if the project were a traded asset. Assuming that markets are efficient, purchasing the project at this price guarantees a zero NPV, and the expected return of the project will be exactly the same as its risk-adjusted discount rate μ . As a result, the mean of the project's returns is exogenously defined.

Step 2

The standard deviation of the returns, or volatility of the project, can be estimated from a Monte Carlo simulation of the project returns. In this process, key project uncertainties are entered as simulation input variables in the project cash flow *pro forma* worksheet, so that each iteration of a simulation of the worksheet provides a new set of future cash flows c_i , $i = 1, \dots, n$, from which a new project value v_1 at the end of the first period is computed from (2):

$$v_1 = \sum_{i=1}^n \frac{c_i}{(1 + \mu)^{i-1}}.$$

Then a sample of the random variable z can be determined using the relationship

$$z = \ln\left(\frac{V_1}{\bar{V}_0}\right), \quad (3)$$

where $\bar{z} = E(z)$ is the mean of the distribution of the project returns between Time 0 and Time 1. The estimate of the standard deviation of z , denoted as s , is obtained from the simulation results. The project volatility σ is then defined as the annualized percentage standard deviation of the returns and is estimated from the relationship $s/\sqrt{\Delta t}$, where Δt is the length of the period in years used in the cash flow *pro forma* worksheet. If the time period between V_1 and \bar{V}_0 is one year, then $\sigma = s$.

Step 3

With the project volatility determined as indicated above, and given the initial expected project value \bar{V}_0 , a binomial lattice can be constructed to model the stochastic process for project value. The volatility for each time period in the binomial lattice is $\sigma\sqrt{\Delta t}$, where Δt is the time period used in the lattice. This is the approach illustrated by CA (2001).

In contrast to the CA approach, we use a binomial tree and express the project value in terms of a more

basic variable: the project cash flows. To do this, we use the cash flow payout rate, $\delta_i = \bar{C}_i/\bar{V}_i$, to calculate the cash flows that are paid out at the end of each time period as a function of the project value. We assume that the cash flows will vary over time, reflecting the uncertainty in the project value, but that they will remain a constant fraction of the residual value of the project in each time period. These cash flows ($C_{i,j}$) will therefore be a function of the project value and the stochastic process that drives the binomial model. The primary advantage of this approach is that it provides greater flexibility in the modeling of the real options of the project.

To obtain the cash flows, we begin by building the tree of pre-cash flow payout values. These values are calculated according to the following equations, where the superscripts u and d correspond to the up and down state values and the state subscript is suppressed:

$$V_i^u = (V_{i-1} - V_{i-1} \delta_{i-1})u$$

$$V_i^d = (V_{i-1} - V_{i-1} \delta_{i-1})d.$$

The logic of this relationship should be transparent. V_{i-1} is the value of the project in the previous state, and $C_{i-1} = V_{i-1} \delta_{i-1}$ is the cash flow paid out at the end of the period, which reduces the project value in the subsequent states.

There are no cash flows in the initial period ($i=0$), since the project has not yet been initiated, so $\delta_0 = 0$. For $i=1$, $V_1^u = uV_0$ and $V_1^d = dV_0$. For all subsequent periods, the cash flow payout rate is assumed to be constant across states in each period but variable in time, so the cash flows in each period are a fixed proportion of the value of the project in that period and state, as noted above. That is,

$$\delta_i = \frac{\bar{C}_i}{\bar{V}_i} = \frac{C_{i,j}}{V_{i,j}} \quad \forall j. \quad (4)$$

Therefore, the discounted cash flow in each period/state is simply given by

$$C_{i,j} = \frac{V_{i,j} \delta_i}{(1+r)^i}. \quad (5)$$

Thus, (5) provides the branch values in each chance node of the binomial tree. Since risk-neutral probabilities are being used, these cash flows are discounted

at the risk-free rate to arrive at the present value of the project at Time $i=0$.

The use of project cash flows in this approach provides a greater level of detail in modeling the operation of the project and the effects of managerial decisions. For example, these cash flows could “ramp up” over the early years of a project as sales are forecasted to grow and decrease at an increasing rate at the end of the project life-cycle. As another example, a model of the development of an oil field could show “lumpy” increases in production as new wells are added, and then show a decrease in production that would follow a decline curve. The model allows simple abandon options to be included in the tree and expansion and contraction options that can be modeled as percentage changes in the underlying cash flows. For example, the option to sell a half interest in the project could be modeled as a 50% reduction in subsequent cash flows, or the option to expand operations could be modeled as a percentage increase in cash flows.

The use of these cash flows, rather than project values, allows the easy use of decision trees rather than binomial lattices to evaluate project options. As a result, the evaluation of real options can be carried out conveniently using “off-the-shelf” decision tree software and allows options to be included in the models using decision nodes that are a natural part of this problem representation.

5. An Example Problem

We illustrate this approach to the evaluation of real options by solving for the value of an oil production project using commercially available decision analysis software, DPL™. While a decision tree representation in DPL™ does not take advantage of the recombining feature of binomial lattices and thus results in larger trees than necessary, it is a convenient and flexible modeling tool that provides a simple and intuitive visual interface.

The example project has estimated reserves of 90 million barrels, and the initial production level of 9 million barrels declines by 15% per year over its 10-year operating life. The variable operating cost starts at \$10 per barrel in Year 0 and grows at 2% per year. Oil price starts at \$25 per barrel and grows at 3% per year. There is also a \$5 million per year fixed

Table 1 Base Case Expected Cash Flows for the Project

Year	0	1	2	3	4	5	6	7	8	9	10
Remaining reserves		90.0	81.0	73.4	66.8	61.3	56.6	52.6	49.2	46.3	43.9
Production level		9.0	7.7	6.5	5.5	4.7	4.0	3.4	2.9	2.5	2.1
Variable op cost rate		10.2	10.4	10.6	10.8	11.0	11.3	11.5	11.7	12.0	12.2
Oil price		25.8	26.5	27.3	28.1	29.0	29.9	30.7	31.7	32.6	33.6
Revenues		231.8	202.9	177.6	155.5	136.2	119.2	104.4	91.4	80.0	70.0
Production cost		(96.8)	(84.6)	(74.0)	(64.8)	(56.9)	(50.0)	(44.0)	(38.8)	(34.3)	(30.4)
Cash flow		135.0	118.3	103.6	90.7	79.3	69.2	60.4	52.6	45.7	39.6
Profit sharing		(33.7)	(29.6)	(25.9)	(22.7)	(19.8)	(17.3)	(15.1)	(13.1)	(11.4)	(9.9)
Net cash flows		101.2	88.7	77.7	68.0	59.5	51.9	45.3	39.4	34.3	29.7
PV of cash flows	404.0	444.5	377.6	317.7	264.0	215.6	171.7	131.8	95.1	61.3	29.7
Cash flow payout rate		0.228	0.235	0.245	0.258	0.276	0.302	0.344	0.414	0.559	1.000

cost that is not shown in the table. The appropriate risk-adjusted discount rate is assumed to be 10% per year, and the risk-free rate is 5% per year. We initially determine the expected value of the future cash flows, which are shown in Table 1. All values are in millions of dollars.

The Year 0 present value of the expected cash flows is \$404.0 million, which was calculated using the risk-adjusted discount rate of 10% per year. This is used as the best estimate of the current market value of the project without options (base case). The required up-front investment is \$180 million, so the project's NPV is \$224.0 million. The project value at the end of each year may be determined using Equation (2), along with the corresponding cash flow payout rate δ_i in each period using (4). For example, the cash flow payout rate in Year 1, δ_1 , is $101.2/444.5 = 0.228$, as shown in Table 1.

In the next step, project uncertainties that may have some correlation with the market are inserted into this deterministic model to perform a Monte Carlo simulation on the project cash flows. We assume that the project has two primary sources of market uncertainty, price and variable operating costs, which follow a GBM stochastic diffusion process with a mean annual rate of increase of 3% and volatility of 15% for the price process and of 2% and 10%, respectively, for the variable costs process. We could have made additional input variables to this model uncertain and included correlations or other relationships among them without any impact on the subsequent computational burden.

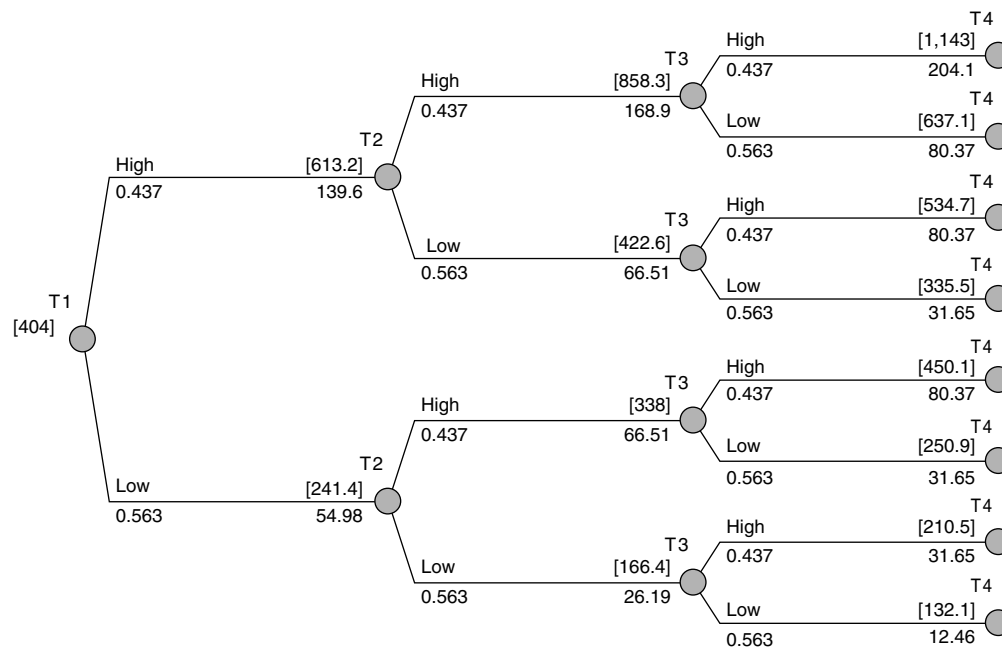
After a large number (e.g., 10,000) of iterations, the Monte Carlo simulation will provide the standard

deviation of the project returns (3) to obtain an estimate of the project volatility, which was determined to be $\sigma = 46.6\%$. This estimate of the project volatility was calculated directly from the simulation as explained earlier, and since the time periods are one year in length in this example, this is the annualized volatility of the project returns. The project volatility may be significantly different from the volatility of the underlying project uncertainties because of the effects of operational leverage. In this example, the impact of price uncertainties on project cash flows may be magnified by the subtraction of operating and fixed costs.

The final assumption is that these returns are normally distributed; consequently, the project values are lognormally distributed and can be modeled as a GBM with constant volatility. The binomial approximation to the GBM process may be modeled using the DPL™ software. The input parameters are the Year 0 value of the project, the volatility σ , the risk-free rate of return r , and the project cash flow payout ratios. The values of u , d , and the risk-neutral probability p are incorporated into the model and computed according to the formulas defined previously. The cash flows in the DPL™ model are computed using (5), and the value of the project is determined by applying the usual procedures of dynamic programming implemented in a binomial tree and discounting the expected cash flows at the risk-free rate of return.

This construction of the tree guarantees that the present value obtained with this model is the same as the one calculated with the spreadsheet, as illustrated in Figure 5, where only the first four of the ten periods

Figure 5 Binomial Tree Model of Value of Project Without Options



of the tree are shown. Tree building can be greatly simplified by developing a standard template for a binomial tree for any given number of time periods. The inputs to the binomial tree can also be linked to a spreadsheet using software packages such as DPL™.

This binomial tree represents the underlying asset and can now be used to evaluate real options. Suppose the project can be divested in the fifth year of its life for a price of \$100 million. The firm might specifically want this option if it is averse to risks later in the project life. Given the binomial tree representation, this option can be evaluated by simply inserting a decision node in Year 5 that models the managerial flexibility that exists in the fifth year of the project.

Additional options can be evaluated by adding the appropriate decision nodes in the tree. For example, suppose the firm can also buy out its partner (assume the partner holds a 25% interest) in Year 5 at a cost of \$40 million. Since the firm already owns 75% of the project, purchasing the remaining 25% represents an increase in value of one-third. A new present value for the project is then computed using the same risk-neutral probabilities, as illustrated in Figure 6, where again, not all nodes are expanded.

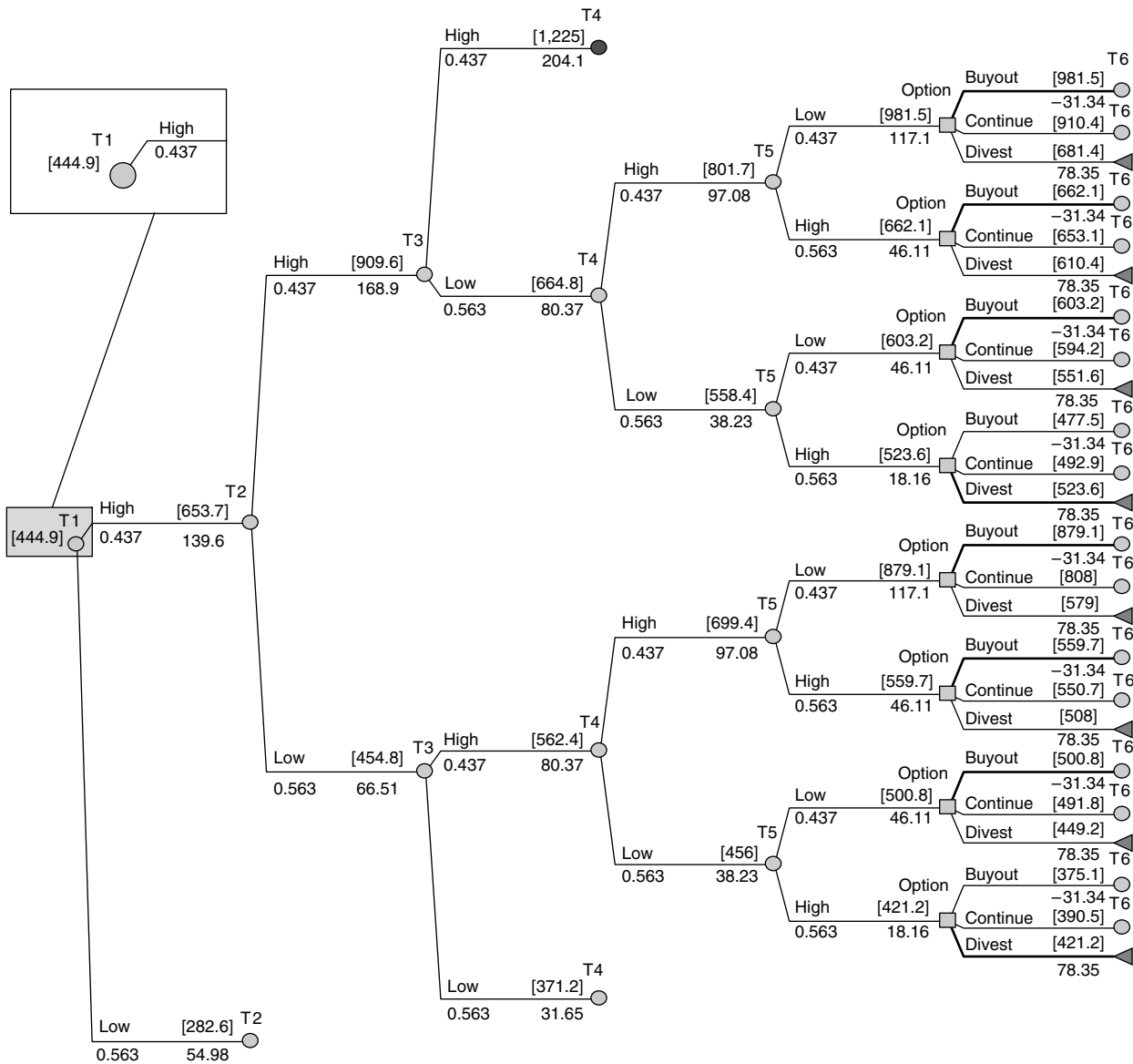
In some of the states the option to abandon by divesting ownership in the project will be exercised,

and in others the buy-out option is exercised. The value of the project with these real options is increased to \$444.9 million, as shown in Figure 6. More options and time periods can be added in a straightforward manner.

As noted earlier, additional market uncertainties could be added to the simulation model and would increase the volatility estimate for the project if not negatively correlated with the other risks. As a result, the value of the options would increase relative to the project base value because of the increase in volatility.

In a manner consistent with the approach Smith and Nau outlined earlier, we can also add private uncertainties to this problem. For example, suppose the oil production in this example is driven by an underlying aquifer, and there is uncertainty about the level at which the oil-water interface exists. When this interface reaches the well, it will begin producing water, and operations will be shut down. This is an example of an uncertainty that has zero correlation with any marketed security. We can model this uncertainty in the decision tree by adding chance nodes in the appropriate time periods in the tree and increasing the probability as time goes on and the limit of oil production is approached. A decision tree for this addition to our model is shown in Figure 7, where we

Figure 6 Value of Project with Option to Buy Out and Divest



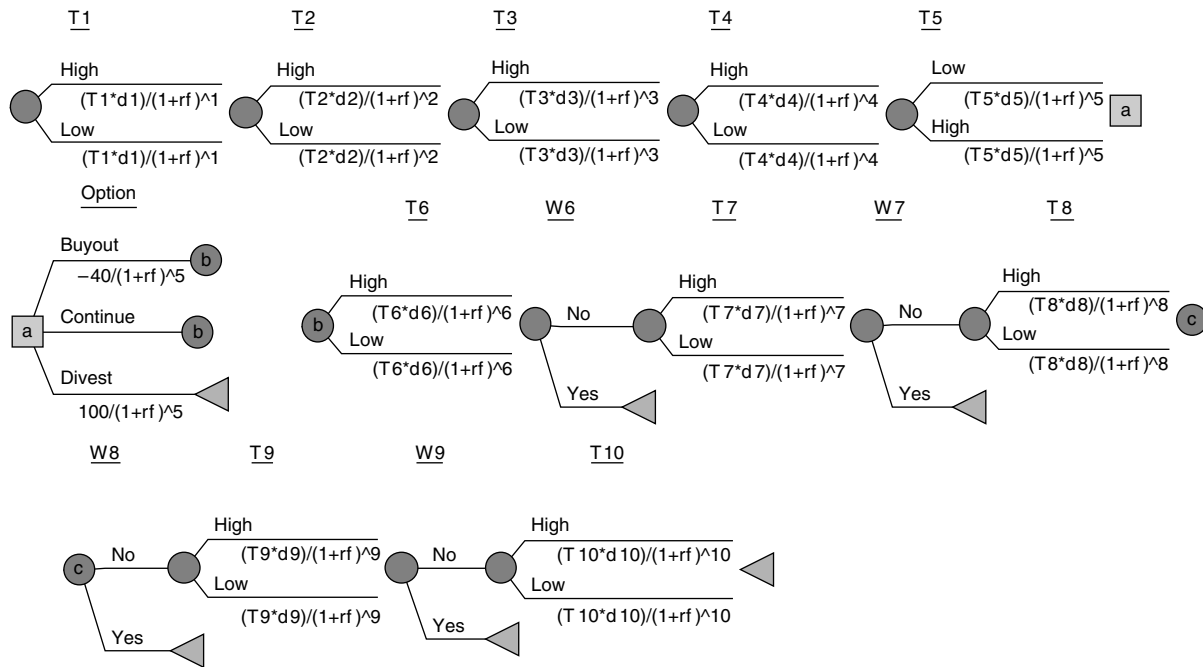
assume that water can only reach the well after five years of operation.

As we would expect, adding this uncertainty reduces the overall value of the project (\$428.0 million, solved tree not shown), since the occurrence of water in the well terminates the cash flows but the exposure to this downside loss is greatly limited by our option to divest. Without this option, the project value would fall to \$397.1 million. This value is easily calculated by

simply removing the decision branch for the abandon option.

Thus far, in considering the private uncertainty, we have assumed the firm is risk neutral. This may be reasonable for a very large firm that has exposure to many such projects. However, a small firm with a limited number of such capital investments may be risk averse, rather than risk neutral. As the cost of the investment increases, a risk-averse firm will have

Figure 7 Project with Real Options and a Private Risk



a decreasing marginal value for the project because it loses the ability to diversify its risks. The risk aversion of such a firm can be modeled by assessing its utility function. For this example, we assume the firm's utility function is the exponential form $U(c_0, c_1, \dots, c_T) = -\sum_{t=0}^T \exp(-c_t / RT_t)$, where c_t and RT_t are the cash flows and risk tolerances, respectively, in each period. We use $RT_0 = \$100MM$ and increase each subsequent risk tolerance over time to reflect the firm's time preference for cash flows, as indicated by a 10% discount rate. An effective risk tolerance for each period can then be calculated as described by Smith and Nau (1995) and entered into the chance nodes for the private risks for the calculation of the certainty equivalent for the project.

The firm's effective risk tolerance is applied to chance nodes for the private risk only, so the risk-neutral view is retained for the chance nodes in the tree that are risk-adjusted by the risk-neutral probabilities. This change in the model results in a drop in value measured by the certainty equivalent to \$400.5 million. Although the value has been further reduced, the risk-averse firm is protected by the abandon option.

6. Discussion

The objective of showing the developments in the previous sections was to illustrate how binomial decision trees can be used to solve real-option problems using the approach suggested by CA (2001). To make this discussion as simple and transparent as possible, we have focused on their basic approach as it is presented in their textbook. However, this approach can be modified to include the use of alternate stochastic processes rather than the GBM, and therefore it provides additional flexibility.

In practice, there are a number of issues that should be considered in an attempt to apply this methodology within a decision-analysis framework. Like all modeling approaches, this framework has its limitations, but it also has some flexibility that should be recognized. We will organize this discussion to focus on the assumptions required by this approach, and on ways in which this model might be extended. As we shall see, the CA approach implemented using binomial decision trees can be viewed as complimentary to the decision analysis approach to solving real-options problems developed by McCardle, Nau, and Smith.

The MAD Assumption

The use of the MAD assumption by CA as the basis for creating a complete market for an asset that is not traded could lead to significant errors, since the approach is based on assumptions regarding the value of the project without options that cannot be tested in the market place. Since identical copies of the project are not freely traded in the market, this should be recognized as a strong modeling assumption to justify the use of risk-neutral pricing for project options.

The choice of the discount rate for the project without options is left to the discretion of the analyst, and the use of the WACC will not be appropriate for all projects even though it is commonly used in practice. Therefore, it is important to realize that the issue of selecting a “risk-adjusted” discount rate for the project is not resolved by this methodology.

Under ideal conditions, the MNS approach avoids this problem by dividing risks into market and private categories and by using information from market-traded commodities (oil prices in the case of Smith and McCardle 1999) or from a correlated stock price (Smith and Nau 1995) to estimate the risk-neutral probabilities for these risks. We agree that this should be done when such market information is available, and in fact it can be incorporated into the CA simulation model as well. For example, the stochastic process for oil price in the example provided in the previous section could easily be specified using market information (e.g., see Schwartz and Smith 2000). It might also be possible to find market replication to approximate the cost process, in which case the appropriate risk-adjusted rate for the project would be the risk-free rate, and this would be logically consistent with the MNS approach.

As a practical matter it may be difficult to identify replicating portfolios of market-traded assets for all market risks in a project. For example, the risks associated with a pharmaceutical company’s new drug might include marketing costs, market size, and price, and it may be impractical to estimate replicating portfolios of market-traded assets for each of them. In such a case, Smith (personal communication, May 2002) suggests estimating the risk premiums for these risks by considering their correlations with the market and effectively estimating their appropriate

“risk-adjusted” discount rates, which would result in a similar discounting approach to the one suggested by CA. Therefore, when used with proper judgment regarding the pricing of market risks, the MNS and CA approaches will use similar modeling inputs.

Market vs. Private Risks

The risks that are included in the simulation model used by CA (2001, Chapter 9)—price, quantity, and variable costs—may all have some correlation with the market and therefore be considered market risks. The use of the simulation model as a basis for estimating the project’s value without options and its volatility should be restricted to include only risks that arguably have some correlation with the market.

CA (2001, Exhibit 10.1) illustrate how to include private risks in their analysis as well, and treat them independently in a manner similar in spirit to the approach suggested by Smith and Nau (1994). That is, these risks are kept separate in their “event tree,” but the solution is still carried out using replicating portfolios at each node. This same problem can be solved using DPLTM, where the discrete approximation to the underlying stochastic process is kept separate from chance nodes representing the private risk. This approach is also illustrated by the incorporation of the private risk associated with the oil-water interface in the example in the previous section.

GBM Assumption

The GBM assumption is a standard one in finance as an estimate of the price or value of a market-traded asset. As indicated earlier, CA provide a rationale and some empirical results to support this assumption as a reasonable one to consider for estimating the future value of a project. However, they also recognize that this assumption may not be appropriate for every project. For example, they discuss the use of the binomial lattice to value options on projects that follow an arithmetic Brownian motion (ABM) in instances when the change in the asset’s value is assumed to be additive rather than multiplicative and project value may go negative.

However, there is considerable flexibility in modeling the underlying stochastic process with a binomial tree. If the primary uncertainty associated with an asset is thought to be mean reverting, as in the

case of oil or other commodity prices, then Hahn and Dyer (2004) show how a binomial tree may be used to approximate such mean-reverting models as the one-factor Ornstein-Uhlenbeck process or the two-factor Schwartz and Smith (2000) process. CA (2001, Chapter 9) also discuss the use of a mean-reverting stochastic process within this framework.

We have illustrated the use of simulation to estimate the volatility associated with a project by calculating its value at the first period only and then assuming that it remains constant over the life of the project, as required by the GBM assumption. That assumption could be verified by calculating the volatility at other time periods during the simulated project life using Equation (3), modified to adjust for the appropriate time period. This would be especially relevant if some of the risks in the simulation model were changing over its life in idiosyncratic ways. For example, if the production rate were decreasing over time, the volatility might be decreasing as well. Other uncertainties may not occur until several time periods have passed, such as those associated with a planned investment decision or new product introduction, or they may even be modeled as jump processes.

This heteroskedasticity could be incorporated by changing the volatility in the binomial tree at the appropriate time periods, which would be implemented by corresponding changes in the values of u , d , and the risk-neutral probability p in these time periods. CA (2001, p. 342) recognize this possibility and note that the stochastic process could be modeled with a binomial tree rather than with a binomial lattice. While it may be possible to develop a recombining lattice with changing volatility over time, this introduces additional complexity into the calculation of the probabilities on the branches. It is relatively straightforward, however, to model a heteroskedastic process using the decision tree approach that we have illustrated.

The obvious alternative to the CA approach is to use the model and distributions from the Monte Carlo simulation to build a traditional decision tree with chance nodes for each uncertainty in each period and value the options in the problem without using the GBM approximation or one of the extensions mentioned above. This may lead to a more complex model and would require the estimation of a set of

conditional probability distributions for the uncertainties in each period where they appear. If the uncertainties were correlated, this approach would become even more challenging. But if one were careful about exploiting the recombining nature of the resulting trees, it could still be manageable.

The representation of the individual market risks with separate chance nodes might provide additional insights into the way the optimal exercise strategies for the options depend on a key uncertainty, and this might be lost when these uncertainties are combined into a single stochastic process using the CA approach. The choice of one approach or the other should depend, we suggest, on both the nature of the problem and the preferences of the modeler.

Binomial Lattice vs. Binomial Trees

We have discussed how binomial trees with risk-neutral probabilities may be used to provide discrete time approximations to the stochastic processes that are often used in the valuation of real options. While this approach is suggested by CA, they emphasize the use of binomial lattices and replicating portfolios. We believe that most decision analysts—and most managers without technical training in real options—would find a problem representation based on binomial trees to have more intuitive appeal.

Even for a simple model such as the one illustrated in the previous section, the decision tree very quickly becomes large. In most practical problems the complexity of the decision tree will be such that full visualization will be impossible. However, even large problems with literally millions of endpoints for the tree can be solved using this approach. Brandão (2002) provides an example of the application of this methodology to the evaluation of options associated with a highway project in Brazil that includes 20 time periods and several different options, resulting in a decision tree with 2×10^9 endpoints that is solved within practical computational times.

If only the expected value of a project is needed, it is not necessary to expand the binomial tree beyond the point at which the last option is introduced as a decision node, since the expected value of this expansion is known at that point. This is illustrated in Figure 4, where the binomial chance node for the third period is not expanded if the decision to invest in the

new plant is made. This could provide some computational efficiency in some applications.

While an n period recombining binomial lattice has a total of $(n + 1)(n + 2)/2$ nodes, an equivalent binomial tree has $2^{n+1} - 1$ nodes, which represents a significant difference for large values of n . Therefore, we conducted a simple comparison of the computational performance of a real options problem modeled with the binomial tree versus the binomial lattice. The problem we selected was the example problem used by Copeland and Tufano (2004) and solved using a binomial tree with $n = 3$ time periods in our earlier example.

We created a binomial lattice to solve this problem using a VBA code and compared its performance to the corresponding binomial tree representation solved using DPL™ (version 6). While other commercial decision tree software, such as PrecisionTree™, could be used for example problems in a classroom setting, we believe that the influence diagram interface in DPL™ is useful for modeling problems of realistic size using this approach. We made no effort to optimize the computational efficiency of the DPL™ software, and simply used the default settings.

According to Hull (2003), in practice solving a binomial lattice with $n = 30$ usually gives reasonable results, so we used this as the upper limit for our range. The results were obtained with an IBM T40 laptop computer using a 1.5 GHz processor and 256 K RAM, and are shown in Figure 8.

As indicated in Figure 8, a well-constructed lattice is much more computationally efficient, which may be very important in large problems or when a high

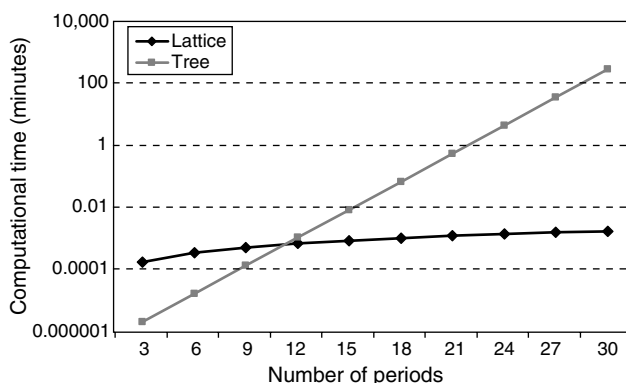
degree of accuracy in the estimate is required. However, the binomial tree is certainly a practical computational tool for $n = 20$ periods and could even be used for larger numbers of time periods, up to approximately the $n = 30$ periods suggested by Hull. A characteristic of the binomial method is that the convergence is not smooth and oscillates around the true value (Clewlow and Strickland 1998, p. 20). For this reason it may be desirable to make several computer runs with binary decision trees of different time periods and average the results. On the other hand, estimates of value in real-options problems may not require the same accuracy that is typically demanded when using lattices to value financial options.

The lattice also provides a representation of the problem that is visually more compact. The optimal exercise decisions can be indicated by shading or formatting values shown in the lattice, and it may be easier to see thresholds, e.g., exercise if the value exceeds some specific number, in the lattice. However, binomial lattices do become complex when dealing with multiple uncertainties, “path-dependent” uncertainties or payoffs, and complex options. These problems can be handled more conveniently with binomial trees. For example, compound options can be modeled simply by adding additional decision nodes to the binomial trees.

According to Triantis and Borison (2001), the choice of a binomial lattice or tree structure by analysts in practice often reflects the background of the individual as well as the complexity of the project being evaluated. Binomial lattices are typically used by those with finance training who are looking at relatively straightforward investment problems.

Perhaps a more relevant comparison of the computational efficiency of the binomial tree based on the CA approach would be with the probability tree required by the MNS method. If there is only one market uncertainty in the corresponding tree, and a trinomial chance node is created with estimates of high, medium, and low outcome values, for example, then after 10 periods it would contain 88,573 nodes, compared with 66 for the binomial lattice and 2,047 for the binomial decision tree. Of course the use of the trinomial chance nodes would provide more precision in the estimation of the stochastic process associated with the risk, so a smaller number of periods might

Figure 8 Comparison of Computational Efficiency: Tree vs. Lattice



be used. If there were two market uncertainties in the problem, as in the example in the previous section, the MNS probability tree with trinomial chance nodes would contain almost 4 billion nodes after 10 periods. In general, it would contain $1 + \sum_{i=1}^n (x^m)^i$ chance nodes, where x is the number of branches at each node, m is the number of market risks, and n is the number of periods. A seasoned analyst would never try to build such a tree and would find ways to trim it to a manageable size. For example, Smith and McCardle (1998, 1999) discuss the use of dynamic programming formulations and lattices in such settings. Nevertheless, it should be clear that the MNS approach will generate very large “dream trees” as well and cannot be applied naively to projects on a period-by-period basis.

In practice, we think that the decision analyst should be aware of the trade-offs between the use of the binomial tree and the binomial lattice to model real-option problems and recognize that there may be situations in which one or the other would be preferred. Similar considerations would apply to the use of the MNS approach as well.

Summary

We have shown an approach for solving real-option valuation problems with decision analysis methods that is consistent with finance-based methods used in practice. This approach provides a straightforward yet flexible way to implement real-option valuation techniques using off-the-shelf decision analysis software. Additional computational efficiencies may be obtained by using specially coded algorithms to solve binomial lattices, although at the cost of having to forgo the simple user interface offered by decision tree programs such as DPL™ and the advantage of visual modeling and a logical representation.

The CA approach can be used to create models that are consistent with the ideas developed by MNS. The primary difference between these two approaches is in the treatment of the market risks in the models. CA suggest reducing them to one stochastic process by focusing on their impacts on cash flows. MNS model these individual risks in each time period. This suggests that the CA approach might be an appropriate choice if there are several market risks and several time periods in a model, whereas the MNS approach

may be the preferred approach if the number of market risks is limited, as in oil and gas exploration. Individual modeling skills and preferences would also be a major consideration.

We agree with Triantis and Borison (2001) that there should be a convergence of real-option evaluation models between finance and decision analysis. In their recent summary article, Smith and von Winterfeldt (2004) also call for more research on the links between decision analysis and finance. The recognition of the similarities between the use of binomial decision trees and the use of binomial lattices for solving real-option problems offers a rich opportunity for further research.

Our comparisons between the CA and MNS approaches have been based on observations and modeling experiences rather than on a rigorous theoretical analysis, and we acknowledge that more could be done to explore these ideas. Likewise, our computational comparisons were merely suggestive of more rigorous work that could be done to investigate the computational properties of these methods.

Based on our experience in modeling ABM and GBM processes, we have also developed a binomial decision tree approach that can be applied to model mean-reverting stochastic processes (Hahn and Dyer 2004). In this spirit there may be more to be gained by reviewing other work on binomial lattices that has appeared in the finance literature and adapting some of these models into a decision analysis framework.

All of the spreadsheets and DPL™ models for the example problems in this paper are available in the Online Supplements section of the *Decision Analysis* web page.

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