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Alternative Approaches for Solving Real-Options Problems

(Comment on Brandão et al. 2005)

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Brandao et al. (2005) describe an approach for using traditional decision analysis tools to solve real-option valuation problems. Their approach calls for a mix of discounted cash flow analysis and risk-neutral valuation methods and is implemented using Monte Carlo simulation and binomial decision trees. In this note, I critique their approach and discuss some alternative approaches for solving these kinds of problems. My criticisms and suggestions concern implementation issues as well as more fundamental issues. On implementation, I discuss the use of binomial lattices instead of trees, and alternative methods for estimating volatilities. More fundamentally, I discuss alternative approaches that rely entirely on risk-neutral valuation and model the uncertainties in the problem more directly.

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1. Introduction

The real-options approach to evaluating investments dates back to Myers (1977, 1984), who suggested using techniques like those used to value put and call options on stocks to value real (nonfinancial) investments where management can "exercise options" to adapt strategies during the course of the project. Some have taken this charge quite literally and suggest using the Black-Scholes formula for valuing put-and-call options on a stock to value real projects (see, e.g., Luehrman 1998). Applying this formula in real-options applications requires drawing an analogy between the project and a put or call option: For example, starting the project is like exercising a call option with a strike price equal to the capital investment required for the project. Of course, decision analysts have long modeled options or "downstream decisions" using decision tree models. While options may be familiar to decision analysts, real-options analysis typically uses a risk-neutral valuation procedure that is less familiar to decision analysts. The risk-neutral valuation procedure incorporates risk premiums by risk-adjusting probabilities rather than risk-adjusting discount rates or determining certainty equivalents using a utility function.

Brandão et al. (BDH) (2005) describe an approach for solving real-options problems that builds on the ideas of Copeland and Antikarov (2001, hereafter C&A) and uses traditional decision analysis tools. Specifically, BDH propose a three-step process:

(1) Calculate the expected net present value (NPV) of the project without options using a deterministic discounted cash flow (DCF) analysis based on a risk-adjusted discount rate.

(2) Estimate the volatility of the value of the project without options using this discounted cash flow model and a Monte Carlo simulation that describes the uncertainty in the project cash flows.

(3) Build a binomial tree that approximates a geometric Brownian motion approximation of the uncertainty in the value of the project without options over time and incorporate options in this tree. In this step, values are calculated using risk-neutral valuation, that is, calculating expected NPVs using risk-neutral probabilities and discounting at the risk-free interest rate. In Steps 1 and 2, BDH follow C&A exactly. In Step 3, they differ from C&A in recommending use of a

binomial decision tree rather than a binomial lattice. Like earlier real-options work, the BDH and C&A approach can be interpreted as building on the analogy with an option on a stock: The first step provides an estimate of the value of the underlying stock and the second step estimates the volatility of this stock. Project cash flows are modeled as a time-varying dividend stream paid by the stock. The numerical methods in Step 3 of the process are quite similar to those used to value American or Bermudan put-and-call options on a stock (those that allow early exercise), although BDH contemplate somewhat more general kinds of options.

In this paper, I critique BDH's approach and discuss some alternative approaches for solving real-options problems. Although I focus on areas where I disagree with BDH's proposal, there are also many points of agreement that should be emphasized. First, above all, we agree that it is important to model the resolution of uncertainty over time, and options that can be exercised as information is gathered. The values, strategies, and insights derived from these dynamic models may be significantly different and richer than those generated by models that do not capture these dynamics. Second, we agree that it is important to recognize the relationship of the project to the market and the implications this has for values and optimal strategies. Again, the values and insights derived from such analyses may be significantly different from those that use a constant corporate discount rate and/or utility function to calculate expected NPVs or certainty equivalents for all projects. Third, we agree that the marriage of simulation, decision tree, and/or dynamic programming models with risk-neutral valuation techniques provides a very fruitful approach for modeling dynamics and incorporating market information into project evaluations.

My criticisms of BDH's approach concern specific implementation issues as well as more fundamental issues. To be constructive as well as critical, I suggest alternative approaches that address these concerns. In §2, I consider the use of binomial lattices in Step 3 of BDH's procedure and compare the lattice approach to BDH's binomial tree representation using BDH's oil production problem as an example. In §3, I show that BDH's (or C&A's) proposed procedure for calculating volatilities often overstates the actual uncertainty in the cash flows and, hence, will overstate the value of many options. In §4, I turn to more fundamental issues and recommend using a fully risk-neutral approach to value projects rather than using a mix of discounted cash flow and risk-neutral methods. The fully risk-neutral approach is, I believe, better grounded in theory and leads to a single coherent valuation model that can be used to value projects with and without options. In §5, I discuss the use of the value of the project without options as the underlying uncertainty and argue that, although this approach is potentially useful in some cases, it is not broadly applicable. In this section, I consider more general techniques that model the underlying uncertainties directly, focusing on the use of Monte Carlo methods (e.g., Longstaff and Schwartz 2001) for solving real-options problems. In §6, I summarize and conclude.

BDH's paper contains numerous caveats and qualifications; many of my criticisms and suggestions elaborate on points they mention in their paper. In critiquing their approach and discussing alternatives, my goal is to help decision analysts better understand these methods so that they can improve their modeling of dynamic decision problems.

2. Binomial Lattices vs. Binomial Trees

BDH discuss the pros and cons of using binomial trees versus using binomial lattices to model real-options problems, and recommend the use of binomial trees. They recognize that binomial trees are more "computationally intensive" than binomial lattices, but argue that binomial trees are "simpler and more intuitive" (see BDH's abstract). Although decision trees may be more familiar to decision analysts, I think that when lattices can be used, they are both simpler and more intuitive than trees. In this section, I will present a lattice model of BDH's oil production example so readers may compare the two approaches. Luenberger (1998) and Hull (1997) provide more detailed introductory discussions of lattices.

Figure 1 shows a lattice model of the BDH oil production example implemented as an Excel spreadsheet; this spreadsheet is available from the journal's

Figure 1 **A Lattice Model of BDH's Oil Production Example**

Assumptions

Initial Value	404.0	u	1.593607
σ	46.6%	d	0.627507
Δt	1	$r\Delta t$	0.05
r	0.05	p	0.437

Cash flow payout ratios

0	1	2	3	4	5	6	7	8	9	10
0.00000	0.2277	0.2350	0.2446	0.2577	0.2759	0.3054	0.3436	0.4144	0.5591	1.0000

Values without options (then-current $million)

0	1	2	3	4	5	6	7	8	9	10
404.0	643.8	792.4	966.0	1,162.9	1,375.6	1,587.3	1,764.7	1,845.9	1,722.6	1,210.4
	253.5	312.0	380.4	457.9	541.7	625.0	694.9	726.9	678.3	476.6
		122.9	149.8	180.3	213.3	246.1	273.6	286.2	267.1	187.7
			59.0	71.0	84.0	96.9	107.7	112.7	105.2	73.9
				28.0	33.1	38.2	42.4	44.4	41.4	29.1
					13.0	15.0	16.7	17.5	16.3	11.5
						5.9	6.6	6.9	6.4	4.5
							2.6	2.7	2.5	1.8
								1.1	1.0	0.7
									0.4	0.3
										0.1

Cash flows (then-current $million)

0	1	2	3	4	5	6	7	8	9	10
0.0	146.6	186.2	236.3	299.7	379.5	480.0	606.3	764.9	963.1	1,210.4
	57.7	73.3	93.0	118.0	149.4	189.0	238.8	301.2	379.2	476.6
		28.9	36.6	46.5	58.8	74.4	94.0	118.6	149.3	187.7
			14.4	18.3	23.2	29.3	37.0	46.7	58.8	73.9
				7.2	9.1	11.5	14.6	18.4	23.2	29.1
					3.6	4.5	5.7	7.2	9.1	11.5
						1.8	2.3	2.9	3.6	4.5
							0.9	1.1	1.4	1.8
								0.4	0.6	0.7
									0.2	0.3
										0.1

Values with options (then-current $million)

0	1	2	3	4	5	6	7	8	9	10
444.9	686.4	848.9	1,060.4	1,333.1	**1,667.6**	1,587.3	1,764.7	1,845.9	1,722.6	1,210.4
	296.7	347.5	412.5	501.8	**632.4**	625.0	694.9	726.9	678.3	476.6
		175.8	191.1	206.1	**224.8**	246.1	273.6	286.2	267.1	187.7
			125.7	128.1	***123.2***	96.9	107.7	112.7	105.2	73.9
				108.2	***109.1***	38.2	42.4	44.4	41.4	29.1
					103.6	15.0	16.7	17.5	16.3	11.5
						5.9	6.6	6.9	6.4	4.5
							2.6	2.7	2.5	1.8
								1.1	1.0	0.7
									0.4	0.3
										0.1

website. Although BDH's Figures 6 and 7 show only a very small portion of the full binomial tree for the problem, Figure 1 provides a complete description of all possible cash flows and values. The top part of the spreadsheet shows the model's assumptions; these are exactly as given by BDH. The initial value is assumed to be \$404.0 million, the volatility σ is 46.6% per year, the time step Δt is one year, and the risk-free rate r is 5% per year. The up multiplier u is given by $\exp(\sigma \Delta t) = \exp(46.6\% \text{ per year} \times 1 \text{ year}) = 1.594$, and the down multiplier d is $1/u = 0.6275$. The probability of an up move p is $(1 + r\Delta t - d)/(u - d) = 0.437$. The cash flow payout ratios δ_t are as discussed by BDH and shown above the matrix in Figure 1.

The top lattice in Figure 1 shows the possible values of the project without options. At each point in the lattice the value may go up with probability $p = 0.437$ or down with probability $1 - p$. Going up corresponds to staying in the same row in the lattice and going down corresponds to moving down one row. If we let $V_{t,j}$ denote the value of the project without options in period (column) t and state (row) j, following BDH, the up and down values in period t are given in terms of the previous period values by

$$V_{t,j} = V_{i-1,j}(1 - \delta_t)u$$
$$V_{t,j+1} = V_{i-1,j}(1 - \delta_t)d.$$

The key feature of the lattice model is that an up move followed by a down move leads to the same value as a down followed by an up; this follows from $d = 1/u$. Thus, the trees in a lattice model recombine rather than explode into a "bushy mess."

The next lattice in Figure 1 shows the possible project cash flows. As discussed by BDH, these cash flows are assumed to be the product of the payout ratios (δ_t) and the values given in the first lattice. The first four years of these cash flows correspond to the cash flows shown in Figure 6 of BDH, although the cash flows in BDH's Figure 6 are stated in present value terms (discounting to period 0 at the risk-free rate r) rather than the then-current terms used in Figure 1.

The third lattice in Figure 1 shows the dynamic programming rollback values for determining optimal strategies and calculating the corresponding values. When there are no options, the period-t state-j value $s_{t,j}$ is given as

$$s_{t,j} = c_{t,j} + \frac{1}{(1+r)}(p s_{t+1,j} + (1-p)s_{t+1,j+1}),$$

so the present value at period t in state j ($s_{t,j}$) is the cash flow received in that period plus the discounted expected value in the next period. To incorporate options, we replace this formula with an expanded version that reflects the options available in a given period. In this example, BDH consider three alternatives at the end of year 5: continuing as before, buying out the partner's 25% share for \$40 million, or selling your share for \$100 million (divesting). In year 5, the rollback values are given by

$$s_{t,j} = \max\left\{ \begin{aligned} & c_{t,j} + \frac{1}{(1+r)}(p s_{t+1,j} + (1-p)s_{t+1,j+1}), \\ & c_{t,j} - \$40 + \frac{4/3}{(1+r)}(p s_{t+1,j} + (1-p)s_{t+1,j+1}), \\ & c_{t,j} + \$100 \end{aligned} \right\} \tag{1}$$

where the three terms in the maximum correspond to the values of three options available. For example, in the second case you get the period's cash flows, pay \$40 million, and then get 4/3 of the expected future value (100% interest rather than 75% interest in the property). The optimal strategies are indicated in the lattice; **bold** indicates that buying out the partner is optimal and ***bold italic*** means divesting is optimal. Other options or options in other periods could be incorporated in a similar manner. The final value and optimal strategies calculated using the lattice model are identical to those given by BDH's decision tree. The present value of \$444.9 million suggests that the option to buy out or divest is worth \$444.9 − \$404.0 = \$40.9 million.

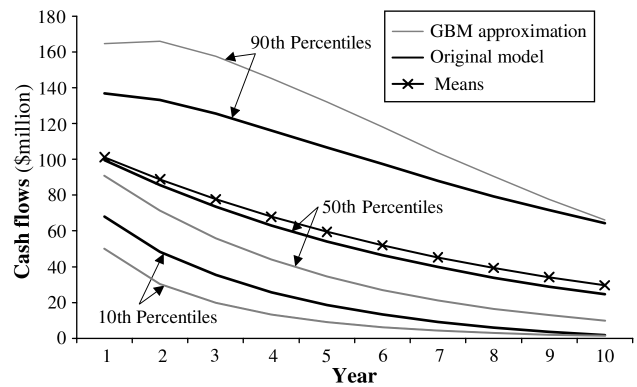
The lattice framework is admittedly less general than the decision tree framework. In order for the tree to recombine, you must be able to identify states—here values of the project without options—such that future probabilities and cash flows depend on the current state, but not on the path taken to reach that state. However, when you can build a lattice (BDH's framework ensures that you can), the lattice provides a much more compact representation than the corresponding decision tree. With a binomial lattice, the number of endpoints is equal to the number of periods n (=11 in the example, counting period 0) and the total number of nodes in the lattice is $n(n+1)/2$ (=66). Models this size can easily be implemented in a spreadsheet, and it is easy to build larger models by simply copying and pasting in Excel; no sophisticated programming or special-purpose software is required. In contrast, the binomial tree has 2^{n-1} (=1,024) endpoints and $2^n - 1$ (=2,047) nodes and is larger still when you include decisions; problems this size require professional decision tree programs. The differences in model sizes become much more pronounced if you consider more periods, either by considering a longer time horizon or finer time steps. Hull (1997, p. 206) notes that when valuing financial options, practitioners typically consider 30 or more periods. With 30 periods, the binomial lattice would have 465 nodes, while the corresponding binomial tree would have more than a billion nodes, without considering any decisions. The lattice is still easily manageable in a spreadsheet. As shown in BDH's Figure 9, the binomial tree requires several hours to evaluate using the most powerful professional decision tree program.

The more compact lattice representation also seems easier to understand. There is less redundant information—the binomial tree repeats the same probability in many places and equivalent states are represented many times—and the states are ordered in a natural way. For example, one can see clearly in the lower lattice in Figure 1 that the optimal policy calls for divesting in low-value states and buying out the partner in high-value states; with these particular parameters, it is never optimal to continue. Of course, this same policy is optimal in the binomial tree, but it is harder to see this structure.

3. Estimating Cash Flow Volatilities

As discussed by BDH, their approach to valuing options approximates the stochastic process for value of the project without options with a geometric Brownian motion (GBM) process whose initial value is determined by the DCF analysis (in Step 1) and whose volatility is determined by a simulation based on this DCF analysis (in Step 2). As discussed earlier, this approximation is motivated by drawing an analogy to the standard model of stock prices used in the Black-Scholes model for valuing options on stocks. A second approximation is the assumption that the cash flows generated by the project without options are a constant proportion of its value. This cash flow approximation simplifies the analysis by allowing you to use the value of the project as a state variable that determines cash flows.

Figure 2 **The GBM Approximation of the Cash Flows in the Oil Production Example**



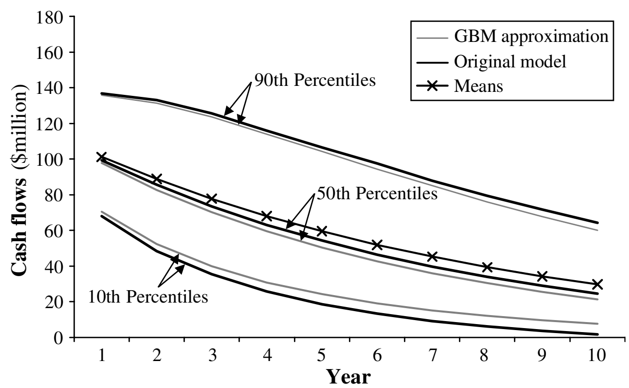
How do these approximations perform in the oil production example? Figure 2 shows the mean and 10th, 50th, and 90th percentiles of the future cash flows using BDH's original cash flow model (in heavy lines) and the same percentiles for BDH's GBM approximation (with light lines). The percentiles for the original model were calculated using BDH's Monte Carlo simulation model with their assumptions. The percentiles for the cash flows in the GBM approximation were calculated using BDH's estimated parameters: The initial value is \$404.0; the volatility is 46.6% per year; the drift is 10% per year (BDH's risk-adjusted discount rate); and the cash flows are generated using the payout rates (the δ_ts) given by BDH. In Figure 2, we see that although the expected cash flows are the same in the original and GBM approximation, the GBM approximation greatly overestimates the uncertainty in the future cash flows: The 90th percentiles are much too high and the 10th percentiles are much too low.

The main problem here is that the volatility is overestimated. In Step 2 of their process, BDH (following C&A) recommend estimating the volatility by tracking $\tilde{z} = \ln(\widetilde{V}_1/\overline{V}_0)$ in a Monte Carlo simulation of the cash flows and taking the volatility of the GBM approximation to be equal to the standard deviation of $\tilde{z}$. Here, $\widetilde{V}_1$ is the period-1 NPV of future cash flows generated in the simulation and $\overline{V}_0$ is the period-0 expected NPV calculated in Step 1 of their process. I do not understand the basis for this recommendation. The standard deviation of $\ln(\widetilde{V}_1/\overline{V}_0)$ would be the appropriate volatility if the values truly followed a GBM (with constant volatility) and $\widetilde{V}_1$ were period 1's *expected* NPV of subsequent cash flows; this volatility would reflect the resolution of a single year's uncertainty and its impact on expectations for future cash flows. In the procedure proposed here, however, $\widetilde{V}_1$ is the NPV of a particular realization of future cash flows that is generated in the simulation, and the standard deviation of $\ln(\widetilde{V}_1/\overline{V}_0)$ reflects the resolution of all future uncertainties. A second concern with this approach is what to do when $\widetilde{V}_1$ is zero or negative and $\ln(\widetilde{V}_1/\overline{V}_0)$ is $-\infty$ or undefined. Negative $\widetilde{V}_1$s occur in about 1 out of 2,000 trials in the oil production example; I suspect these scenarios were simply discarded in BDH's calculations. These zero and negative values are not possible in the GBM approximation.

Although I cannot think of any easy way to convert the volatility that BDH calculate into an appropriate volatility, one fairly simple way to estimate a more appropriate volatility would be to simulate the GBM approximation and search for a volatility for the GBM approximation that matches, as well as possible, the uncertainty in the original model. For instance, in the oil production example, the current NPV of all cash flows ($\widetilde{V}_0$) has mean and standard deviation of \$404 million and \$168 million, respectively. If we assume a volatility of 25.5% per year in the GBM approximation, we find a mean and standard deviation of $\widetilde{V}_0$ of \$404 million and \$168 million, matching those in the original cash flow model. The cash flow percentiles for this approximation are shown in Figure 3, along with those in the original model. The fit here is much better than that shown in Figure 2, but still not perfect.

Using this volatility of 25.5% per year in the lattice model of Figure 1, we find that the project with options is now worth \$423.7 million rather than the \$444.6 found using a volatility of 46.6% per year. The value of the option with a volatility of 25.5% per year is \$423.7 − \$404.0 = \$19.6 million, slightly less than half of the value (\$40.4 million) given by assuming

Figure 3 **An Alternative GBM Approximation**



a volatility of 46.6% per year. Clearly, the volatility estimate makes a difference!

Although our discussion of BDH's volatility estimate has focused on how well the process approximates the current unconditional distribution for cash flows, we should also consider how well the approximation captures future conditional forecasts that determine the optimal exercise decisions. For example, a mean-reverting process may lead to very different policies and values than a GBM process; see, e.g., Smith and McCardle (1999). Moreover, if the underlying uncertainties have a variety of different forms of processes—for example, if oil prices are mean reverting and costs follow a GBM—then the overall project value may not be well approximated by any simple univariate process, such as a GBM or a mean-reverting process. In general, the problem of approximating a high-dimensional stochastic process with a low-dimension (here univariate) summary process is an interesting research problem that goes well beyond the question of how to pick a volatility parameter for a GBM process.

Although this is an interesting research question, ultimately I am not convinced such a univariate approximation is necessary because, as discussed in §4 below, one can instead model the underlying uncertainties—here, oil prices and variable operating costs—directly and approximately solve the model using Monte Carlo techniques. This more direct approach not only avoids the need to approximate the value process, it also allows one to model additional kinds of options. However, if you are going to use this GBM approximation (or another simple univariate approximation), it is important to think carefully about the assumed volatility and to consider how well the approximation matches the uncertainty in, and dynamics of, the actual cash flows.

4. On Risk-Neutral Valuation

BDH and C&A's valuation procedure uses a mix of discounted cash flow and risk-neutral valuation methods. In Steps 1 and 2 of their procedure, they determine the NPV of the project without options and its volatility using a discounted cash flow (DCF) analysis that calculates NPVs using a risk-adjusted discount rate. When valuing options in Step 3, C&A assume that this project without options is a traded security and use this hypothetical security to construct a portfolio that replicates the payoffs of the project with options. BDH carry out an equivalent analysis using risk-neutral valuation, i.e., using risk-adjusted or risk-neutral probabilities and discounting at the risk-free rate.

Rather than using a mixture of DCF and risk-neutral valuation techniques, we could instead use a fully risk-neutral approach where we construct a single, coherent risk-neutral model and use it to estimate the value of the project both with and without options. In this approach, we would risk-adjust the probabilities or processes associated with the uncertainties or stochastic factors in the model (e.g., oil prices and the variable operating costs) and calculate the value of any investment—including the project without the option—by determining its expected NPV using these risk-neutral probabilities or processes and discounting at the risk-free rate. We will briefly review two different ways to justify this fully risk-neutral approach in situations where the project cannot be perfectly replicated by trading securities.

Equilibrium Valuation. First, as is common in the finance literature (see, e.g., Schwartz 1994), we can justify the fully risk-neutral approach using an equilibrium model of asset prices such as that developed by Cox et al. (CIR 1985) to estimate a risk adjustment for the drifts or growth rates of the stochastic processes for the uncertain factors in the model. If we had true drift μ for some process, the risk-neutral drift is given by $\mu^* = \mu - \lambda$, where λ is a risk premium that depends on the correlation of the factor with aggregate wealth and the other factors in the economy. Drawing on Merton's (1973) intertemporal capital asset pricing model (CAPM), researchers often take the risk premium for factor i to be $\lambda_i = \beta_i(r_m - r)$ where r_m is the expected return on the market portfolio, r is the risk-free rate, and β_i is the beta for factor i (given by σ_{im}/σ_m^2 where σ_{im} is the covariance between the factor and the market and σ_m^2 is the variance of the market). The project value given by this equilibrium approach is an estimate of the market value of a project, the value the project would have if it were traded in a market in equilibrium.

Although Schwartz (1994) and others describe this equilibrium approach to risk-neutral valuation as one that can be applied to investments that cannot be perfectly replicated by trading securities, if you read CIR closely (see p. 368), you will note that their model assumes that markets are complete in that any contingent claim can be replicated by a linear combination (perhaps with weights changing over time) of a set of basis claims that are traded. This is, in some respects, like C&A's MAD (Marketed Asset Disclaimer) assumption, which assumes that the project without options is itself a traded stock that can be used to form replicating portfolios. CIR's assumption is stronger in that it assumes that any claim tied to the underlying stochastic factors can be replicated. However, despite its name, the MAD assumption or CIR's stronger and more precise version of it, does not strike me as crazy: If the goal is to estimate the value the project would have if it were traded in a market in equilibrium, then it seems reasonable and consistent to calculate these values by contemplating a world (or economy) where the project and its underlying stochastic factors can be replicated and/or are traded. The equilibrium approach is constructive in that it provides a framework for thinking about and estimating risk premiums for stochastic factors that cannot be replicated.

Applying this equilibrium approach in the oil production example, we could estimate the risk-neutral drift for oil prices by considering prices for long-term oil futures or forward contracts. For example, if the long-term futures prices are not growing or declining, this would suggest that the risk-neutral drift is 0% per year. The risk premium λ_i is the difference between the true drift (assumed by BDH to be 3% per year) and the risk-neutral drift. If we assume that the variable operating costs are uncorrelated with the market portfolio, then this equilibrium approach would suggest that no risk adjustment is required, and the risk-neutral drift would be the same as the true drift of 2% per year.

CIR's equilibrium model also specifies risk-adjusted discount rates that can be used to calculate project values as expected NPVs using the true (rather than risk-neutral) stochastic processes for the underlying factors. However, these risk-adjusted discount rates—unlike the risk adjustments to the stochastic factors in the risk-neutral approach—will vary from project to project and, for a given project, will typically vary with time and with the values of the stochastic factors. For instance, in the oil production example, even if the risk premiums for oil prices and variable operating costs are constant, the equilibrium discount rate for the project without options would change over time as the production rate decreases and prices and operating costs vary.

Decision Analytic Valuation. A second way to justify this fully risk-neutral approach is to use the decision-analytic valuation procedure developed in Smith and Nau (1995). As BDH note, this procedure distinguishes between market risks that can be hedged by trading existing securities, and private risks that cannot. Project values are given by an integrated valuation procedure that, in the case of a risk-neutral decision maker, determines project values by calculating expected values using risk-neutral probabilities for market risks and true probabilities for private risks; all discounting is done at the risk-free rate. In this framework, dependence between market and private risks is captured by assessing true probabilities for the private risks conditional on the contemporaneous market state. The values given by this procedure are justified by an extension of the Fisher Separation Theorem, which shows, given certain utility assumptions, that these values are indifference prices for a decision maker who simultaneously considers investments in securities and the projects: The decision maker would be just indifferent to buying or selling the project for these amounts. In an appendix to this note, we discuss how to handle uncertainties that "fall somewhere in between the notions of private and market risks" (BDH, p. 77) in this framework.

In the oil production example, if we assume that oil prices are market risks with a risk-neutral growth rate of 0% per year and that the variable operating costs are independent of oil prices and are private risks, then the integrated valuation procedure uses the risk-neutral process for oil prices and the true process for costs. If we also assume that the decision maker is risk neutral, then the resulting valuation model will be the same as that given by the equilibrium approach. Thus, these two different lines of argument lead to the same fully risk-neutral valuation procedure.

Results for the Example. As indicated earlier, in this fully risk-neutral approach the value of an investment is its expected NPV given by using these risk-neutral processes and discounting at the risk-free rate. You can calculate these expected NPVs however you want. For a project with embedded options, you might use a decision tree or lattice model. For a project without options, you might use Monte Carlo simulation. In the oil production example, because the cash flows and NPVs are linear functions of these underlying uncertainties, we can calculate the expected NPV of the project without options using a deterministic model based on expected oil prices and variable operating costs, like that shown in BDH's Figure 5. Figure 4 shows a risk-neutral version of this deterministic analysis that assumes a risk-free discount rate of 5% per year and risk-neutral growth rates of 0% and 2% per year for oil prices and variable operating costs, respectively. The result is an NPV of $392 million, compared to an NPV of $404 million given by BDH's analysis.

The difference between these two NPVs reflects fundamental differences in the way the two approaches value future oil production and costs. In BDH's DCF analysis, oil prices grow at 3% per year and are discounted at 10% per year; thus, in net, the present value of future oil production decreases 7% per year. With the risk-neutral procedure, prices grow at 0% and are discounted at 5% and, in net, the present value of future production decreases 5% per year. Consequently, the risk-neutral procedure places higher values on future oil production. However, the risk-neutral valuation procedure also places higher

Figure 4 A Risk-Neutral Version of BDH's Base Case Analysis

Year	0	1	2	3	4	5	6	7	8	9	10
Production Level		9.0	7.7	6.5	5.5	4.7	4.0	3.4	2.9	2.5	2.1
Variable Op Cost Rate		10.2	10.4	10.6	10.8	11.0	11.3	11.5	11.7	12.0	12.2
Oil Price		25.0	25.0	25.0	25.0	25.0	25.0	25.0	25.0	25.0	25.0
Revenues		225.0	191.3	162.6	138.2	117.5	99.8	84.9	72.1	61.3	52.1
Production Cost		(96.8)	(84.6)	(74.0)	(64.8)	(56.9)	(50.0)	(44.0)	(38.8)	(34.3)	(30.4)
Cash Flow		128.2	106.7	88.6	73.4	60.6	49.9	40.9	33.3	27.0	21.7
Profit Sharing		(32.1)	(26.7)	(22.1)	(18.3)	(15.1)	(12.5)	(10.2)	(8.3)	(6.8)	(5.4)
Net Cash Flows		96.2	80.0	66.4	55.0	45.4	37.4	30.7	25.0	20.3	16.3
PV of Cash Flows	392.0	411.6	331.2	263.8	207.3	159.9	120.1	86.9	59.0	35.8	16.3
Cash Flow Ratios		0.2336	0.2415	0.2518	0.2654	0.2842	0.3113	0.3528	0.4233	0.5664	1.0000

values on future costs: The growth rates for costs are not risk-adjusted and the risk-neutral procedure discounts them at 5% per year instead of 10%. In this particular example, the cost effect is dominant and the risk-neutral value is lower than the risk-adjusted value. However, in other examples—such as an exploration play—where the production streams are further in the future, the higher value of future production could be dominant and we might find higher values with the risk-neutral valuation procedure.

In summary, I prefer the fully risk-neutral approach to BDH's mix of risk-adjusted discount rate and risk-neutral approaches for three reasons. First, although the assessment of factor- and/or uncertainty-specific risk adjustments may require subjective judgments of correlations (or betas) that are similar to those required in estimating a risk-adjusted discount rate, I believe that the decomposed, factor-level risk-adjustments are likely to be easier to think about and estimate than an aggregate risk-adjusted discount rate. Second, these factor-specific adjustments can more plausibly be assumed to be constant over time, and the same ones can be used consistently for different projects that involve these same factors. Third, the fully risk-neutral approach leads to the development of a unified and coherent probabilistic model that can be used to value projects with and without options.

5. On the Choice of Underlying Uncertainty

As discussed earlier, BDH's approach to valuing options is based on approximating the stochastic process for the value of the project without options with a geometric Brownian motion (GBM) process; BDH also mention the possibility of using, for example, arithmetic Brownian motion or mean-reverting processes. Various project options are then viewed as if they were derivative securities whose cash flows are derived from the value of this project without options.

Although this approach has the advantage of reducing a potentially complex multidimensional problem to a univariate problem, I am not convinced that many real-options problems can be formulated this way. In the oil production example, the options considered are both examples of what might be called a scale option, the ability to increase or decrease one's interest in the investment in exchange for a cash payment or receipt. In this example, one option is to buy out the partner's 25% share for $40 million. The second option is to sell your 75% share in the project for $100 million. Scale options are easy to evaluate in this framework because the value of the option can be easily determined as a function of the value of the project without options. For example, in the buyout option, the cash flows and values after exercise are given by multiplying the cash flows and values of the project without options in each period by 4/3. C&A also consider exit and scale options in their case study illustrating this approach.

Although scale options are certainly important, it is difficult to value many other kinds of options using the value of the project without options as the underlying uncertainty. For example, it is not clear whether an option to delay the start of a project can be formulated this way. In the oil production example, the evolution of value of the project without options reflects the resolution of uncertainty in costs. In practice, this cost uncertainty might evolve differently—perhaps with less volatility—or not at all, if you are not actually producing at the site; clearly, production would be delayed. In other real-options models, we might be interested in options that depend on particular uncertainties. For example, one might consider signing a contract with a customer that places a cap or floor on the price paid for the oil produced. Alternatively, one might consider the possibility of using enhanced oil-recovery methods that would produce additional oil but with a different cost structure. As I reflect on my experiences in modeling real-options problems, I think these scale options are the only options that I have encountered whose payoffs can be determined as a function of the value of the project without options.

To value more general options, we typically need to use models that consider the evolution of the underlying uncertainties directly. In the oil production example, such a direct model would consider uncertainty in both oil prices and variable operating costs over time. We could formulate this model as a two-dimensional lattice model in which we keep track of both the oil price and the variable operating costs in each period. (See, e.g., Luenberger 1998 for a discussion of two-dimensional lattices.) We could do the same thing, albeit less efficiently, using a two-dimensional binomial or trinomial tree. Although these approaches would suffice for this two-dimensional example, BDH seek general methods that can handle more uncertainties without becoming overly difficult to implement. As BDH point out, both lattices and trees suffer from the "curse of dimensionality" and their size and complexity grows rapidly with the number of uncertainties in the problem. Decision tree modelers and lattice builders are thus often forced to focus their models on a few key underlying uncertainties.

Although there is no "magic bullet" for simplifying these complex problems, the new Monte Carlo methods developed for valuing high-dimensional financial options appear quite promising for real-options problems as well. (See Glasserman 2004 for a comprehensive review of Monte Carlo methods in finance.) In the simulation approach for valuing options, we build a Monte Carlo simulation model that takes into account all of the uncertainties in the problem, which can then be used to calculate expected NPVs for any given exercise policy. If we want to use risk-neutral valuation, then we should use risk-neutral probabilities for the uncertainties and discount at the risk-free rate. We then approximate the optimal exercise policy using one of many different possible methods. Finally, given this near-optimal exercise policy, we calculate the expected NPV of the project using this exercise policy. As these near-optimal exercise policies are feasible, the project value generated by this procedure provides a lower bound on the project value that would be found using a truly optimal policy.

To calculate an optimal exercise policy, at each decision point, we need to examine the expected future NPV (the continuation value) for each alternative, conditioned on the resolution of all uncertainties up to that time. The optimal policy then selects the alternative with the maximum continuation value in a given information state. This is how the roll-back procedure for solving decision trees or dynamic programs works (see, e.g., Equation (1)). There are a variety of ways one can construct near-optimal policies using simulation. In some cases, we can assume a parametric form for the exercise policy and estimate an optimal parameter. For example, if we have a single decision and think that the optimal policy should be of the form "exercise if the oil price is above some threshold," we can run simulations with different threshold oil prices and note the threshold price that leads to the highest expected NPV. This approach will work well if you can identify a good parametric family of exercise policies that has relatively few parameters.

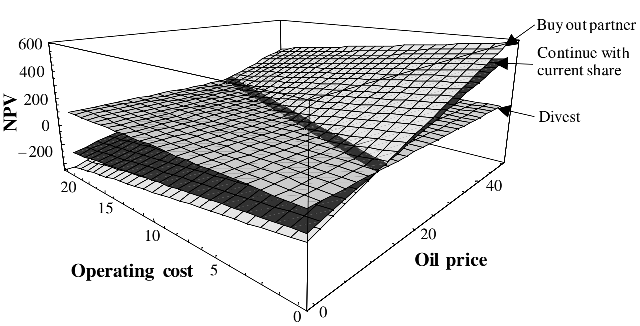
Alternatively, following Longstaff and Schwartz (2001), you can estimate the required continuation values using linear regression and then use these estimated regression equations to determine a near-optimal policy. We will illustrate Longstaff and Schwartz's approach in the oil production example. In this example, the option to adjust partnership interests is exercised in year 5 and we want to estimate the expected continuation values as a function of the oil price and variable operating costs at that time. To do this, we can run a simulation model for the project without options (e.g., assuming that you continue without buying out your partner or divesting), recording the NPV of the cash flows after year 5 (the realized continuation value) and the year 5 oil price and variable operating costs in each scenario. We then run a regression relating the realized continuation values (Y) to the year 5 oil prices (p) and variable operating costs (c). For example, using the fully risk-neutral model discussed in the previous section, in one particular simulation of 10,000 trials we found the following estimated regression equation:

$$\widehat{Y} = -18.49 + 9.77p - 10.06c. \tag{2}$$

Although these linear terms are sufficient in this case, we could easily include additional terms in this regression, for example, nonlinear transformations or powers of p and c, as well as products of these variables. If there were additional uncertainties in the problem that might affect these expected continuation values, you can simply add additional basis functions to the regression model.

The estimated regression equation provides an estimate of the expected continuation value as a function of the year 5 state variables and can be used to determine a near-optimal exercise policy: In any scenario, we calculate the estimated expected continuation value $\widehat{Y}$ from the values of p and c generated in that trial using the regression equation (2). If this $\widehat{Y}$ is less than \$100 million, then you should divest in year 5. If the estimated value of the partner's share ($\widehat{Y}/3$) exceeds the cost to buy it out (\$40 million), then you should buy out the partner. In the other scenarios (i.e., if the estimated $\widehat{Y}$ is between \$100 and \$120 million), you should continue without adjusting the partnership interests. Figure 5 shows these estimated continuation values ($\widehat{Y}$) for the three alternatives as a function of year 5 oil price (p) and year 5 operating cost (c). For scenarios with low costs and high prices, this near-optimal strategy calls for buying out the partner. With high costs or low prices, you should divest. For a relatively small range of prices and costs (the black region in the center of Figure 5), continuing with the current share is recommended. Simulating using this near-optimal policy, we find an expected NPV of \$421 million, compared to an expected NPV of \$392 million without the option to change the partnership structure in year 5.[1]

Figure 5 **Estimated Continuation Values for the Oil Production Example**



Longstaff and Schwartz's method readily generalizes to more complex problems. Although the complexity of the procedure is relatively insensitive to the number of uncertainties in the problem, its complexity grows with the number of decisions and alternatives in the problem in the same way as decision trees. In general, we need to perform this kind of regression analysis to estimate conditional expectations for each alternative of each decision in the model, working backwards from the last decision towards the first. In the oil production example, we only need to do one regression analysis for the continue alternative because the expected continuation value for the divest alternative is a constant, and the expected continuation values for the buyout option are a simple linear function of the continue values. If the buyout option had a more complex impact on the cash flows (suppose, for example, it altered the costs), then we would have to run a second simulation that assumes you buy out your partner in year 5 and then run a regression with these simulations results to estimate the expected continuation value for this alternative. If we had additional decision points, we would need to run additional regressions, working backwards from the last set of decisions. For example, if we had a similar set of buyout or divest options in period 3, we would

[1] In this particular example, because (a) the cash flows are a linear function of oil prices and variable operating costs, and (b) the expected future prices and operating costs are a linear function of current prices, the expected continuation value is in fact a linear function of these two state variables. These exact continuation values can easily be calculated in the spreadsheet. The near-optimal policy thus has the same form as the true optimal policy, and the policies and corresponding values differ only because of sampling errors in the simulations used to estimate the regression equation and final values. The value given by using the exact optimal policy is very close to that given by using this approximate policy.

first determine an approximately optimal policy for period 5 and then use this near-optimal policy when estimating continuation values for period 3.

Longstaff and Schwartz (2001), Tsitsiklis and Van Roy (2001), and Glasserman (2004) discuss these simulation procedures in detail and consider their convergence properties. As these and other authors discuss, in practice there is an element of art in selecting good basis functions to use in these regression models. However, experience in valuing financial options suggests that the procedure is quite robust and will perform well with a wide range of different basis functions.

6. Conclusions

Let us take stock and summarize. We have considered a series of variations on BDH's proposed method for solving real-options problems. The results for the oil production example for each of these variations are shown in Table 1. In §2, we argued for the use of binomial lattices, when applicable, instead of binomial trees. Although the results in this example are identical to those given by a binomial tree, I believe the lattice framework is simpler and easier to understand and apply in these problems; readers may judge for themselves. In §3, I argued that the BDH's volatilities overestimate the uncertainty in many problems, and consequently overestimate the value of many options. If we reconsider the oil production example using a more appropriate volatility of 25.5% per year, the value of the option is slightly less than half the value given by BDH's analysis.

Table 1 Summary of Results for Different Approaches

Model/assumption	Solution technique	NPV of project without options	NPV of project with options	NPV of option
1. BDH approach with $\sigma = 46.6\%/\text{yr}$	Binomial lattice	404.0	444.9	40.9
2. BDH approach with $\sigma = 25.5\%/\text{yr}$	Binomial lattice	404.0	423.7	19.7
3. Fully risk-neutral base case $w/\sigma = 25.5\%/\text{yr}$	Binomial lattice	392.0	411.6	19.6
4. Fully risk-neutral, direct approach	Monte Carlo simulation	392	421	29

Although the variations in §§2–3 are important, they preserve the general framework of BDH's approach. The variations considered in §§4–5 are more fundamental. In §4, I argued that rather than using a mix of risk-neutral and discounted cash flow methods, we should use a fully risk-neutral approach to value the project with, as well as without, options. I believe this approach is better grounded in theory and easier to apply correctly. In this example, the fully risk-neutral approach leads to a lower value, but in other cases we may find higher values. If we use this new NPV as the starting point for a GBM approximation of the value of the project without options, we find that even though the overall values are lower, the value of the option does not change much.

In §5, we consider the use of the value of the project without options as an approximate underlying uncertainty. Although this approximation may be helpful in simplifying the analysis of scale options, to evaluate other kinds of options we typically need to model the project's actual underlying uncertainties. If there are many uncertainties, it may be difficult to model the project using lattices or decision trees, but new simulation methods may be quite helpful in these settings. This simulation approach is quite general—one can consider many different kinds of options and many uncertainties—and the near-optimal policies can be readily interpreted in terms of the project variables. For instance, in the oil production example, the near-optimal policies describe what to do as a function of prices and variable operating costs. Even though these near-optimal policies may be more complex to find, they seem easier to interpret than policies stated in terms of the value of a hypothetical project without options.

Let us also recap some of the key points where BDH and I agree. First, we agree that it is important to model dynamics in decision problems and recognize the value of options associated with investments. Second, we agree that it is important to recognize the impact of the market on project values. Finally, we agree that Monte Carlo simulation, decision-tree, and/or dynamic programming models provide powerful tools for valuing projects and options using risk-neutral valuation methods. Although I have been critical of aspects of BDH's approach, I believe that their approach also contains some useful elements—particularly using decision-tree models with risk-neutral probabilities to model real options—that decision analysts should understand and appreciate.

In critiquing BDH's approach and discussing alternatives, my goal has been to help decision analysts better understand these methods so that they will be better able to model dynamic decision problems. Although some of the methods discussed here—risk-neutral valuation, lattices, and Monte Carlo methods for dynamic programming—may be unfamiliar to many decision analysts, these tools can be quite useful for modeling project dynamics and options. The tools that decision analysts know well—including influence diagrams, decision trees, and probability assessment methods—are also quite helpful for modeling project dynamics and options. In summary, although I disagree with aspects of BDH's proposal, we agree that there is much to be gained from integrating the real options and decision analysis approaches to project evaluation.

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Appendix
On page 77, BDH write:

> In many projects, some uncertainties fall somewhere between the notions of private and market risks. For example, a new drug development project for a pharmaceutical company may not include risks that can be replicated by a traded asset, but the price of the product is clearly a "market risk."

In this appendix, I use this example of a drug development project to illustrate how the integrated valuation procedure can be applied to "uncertainties that fall somewhere between the notions of private and market risks."

First, although the price for the pharmaceutical product is related to the market for the product, it is clearly not a "market risk" in the sense of the integrated valuation framework, which would require that it could be perfectly hedged by trading securities. However, this price might be correlated with the market portfolio represented by, say, the S&P 500. We can capture this dependence in the integrated framework by expanding the valuation model to consider the return on the market portfolio as well as the price for the pharmaceutical product, explicitly modeling the dependence between these uncertainties. However, as we will demonstrate, this expansion need not complicate the model used to evaluate the project.

Now let us be specific and assume that the price p for the product in a given period and the market return m for that period have a bivariate normal distribution with marginal distributions $p \sim N(\mu_p, \sigma_p^2)$ and $m \sim N(\mu_m, \sigma_m^2)$ and correlation coefficient ρ_{pm}. The conditional distribution $p \mid m$ is then $N(\mu_p + \beta(m-\mu_m), (1-\rho^2)\sigma_p^2)$ where $\beta = \rho_{pm}(\sigma_p/\sigma_m) = \sigma_{pm}/\sigma_m^2$ can be interpreted like a beta in the CAPM. If we assume that the return on the market portfolio can be hedged, then m is a market risk. Assuming there are no dividend payments, the risk adjustment for this traded security would typically set the mean of the risk-neutral distribution to r, so in the risk-neutral model $m \sim N(r, \sigma_m^2)$. The price p for the pharmaceutical product, however, cannot be hedged, and thus is a private risk. The integrated valuation procedure uses the conditional probabilities for $p \mid m$, and these would not be risk adjusted in any way.

Now suppose the decision maker is risk neutral and evaluates investments using the integrated valuation procedure. In this case, project values are expected values calculated using risk-neutral probabilities for market risks and true conditional probabilities for private risks. If the project cash flows depend on the market return m only through m's impact on the price p, we can integrate out this uncertainty about m and collapse this expanded model to a simpler model that considers uncertainty in p but not m. However, to arrive the correct project values, when integrating out the uncertainty about m, we must use the risk-neutral distribution for m together with $p \mid m$. The resulting marginal distribution for p is $N(\mu_p + \beta(r-\mu_m), \sigma_p^2)$, where the mean $\mu_p - \beta(\mu_m - r)$ reflects a risk premium of eta $\beta(\mu_m - r)$ for the correlation with the market that is very much like that given by using Merton's intertemporal CAPM to risk-adjust drifts in the equilibrium risk-neutral approach.

Thus, the integrated valuation procedure can accommodate risks that "fall somewhere between the notions of market and private risks" without complicating the evaluation model. To apply this approach in practice, we need to assess the correlation or beta for the uncertainty. As discussed at the end of §4, the judgments involved are similar to those typically used to specify a risk-adjusted discount rate for the project as a whole. However, I suspect that in most cases these judgments would be easier to make for the individual uncertainties in the problem than at the aggregate project level.

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