Dynamic Portfolio Optimization with Transaction Costs: Heuristics and Dual Bounds

(*Online Appendix*)

David B. Brown and James E. Smith

Fuqua School of Business
Duke University
Durham, NC 27708-0120

dbbrown@duke.edu, jes9@duke.edu

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A. Appendix

A.1. Proofs and Detailed Derivations

Proof of Proposition 2.1. The proof is by induction. For the terminal case, $V_T(\boldsymbol{x}_T, c_T, \boldsymbol{z}_T) = U(\mathbf{1}'\boldsymbol{x}_T + c_T)$ is nondecreasing in c_T and concave in $(\boldsymbol{x}_T, c_T)$ because the utility function is assumed to be nondecreasing and concave in wealth. We now assume the result of the proposition holds for period $t+1$ and show that it must also hold for period t.

For monotonicity in c_t: Assume $c_t^1 \leq c_t^2$. Then by the induction hypothesis, for any $\boldsymbol{x}_t$ and $\boldsymbol{a}_t$,

$$\begin{aligned}
W_t(\boldsymbol{a}_t, \boldsymbol{x}_t, c_t^1, \boldsymbol{z}_t) &= \mathbb{E}\left[V_{t+1}\big(\tilde{\boldsymbol{r}}_{t+1} \cdot (\boldsymbol{x}_t + \boldsymbol{a}_t),\ r_f(c_t^1 - \mathbf{1}'\boldsymbol{a}_t - \kappa(\boldsymbol{a}_t)),\ \tilde{\boldsymbol{z}}_{t+1}\big) \mid \boldsymbol{z}_t\right] \\
&\leq \mathbb{E}\left[V_{t+1}\big(\tilde{\boldsymbol{r}}_{t+1} \cdot (\boldsymbol{x}_t + \boldsymbol{a}_t),\ r_f(c_t^2 - \mathbf{1}'\boldsymbol{a}_t - \kappa(\boldsymbol{a}_t)),\ \tilde{\boldsymbol{z}}_{t+1}\big) \mid \boldsymbol{z}_t\right] \\
&= W_t(\boldsymbol{a}_t, \boldsymbol{x}_t, c_t^2, \boldsymbol{z}_t).
\end{aligned}$$

This implies

$$\begin{aligned}
V_t(\boldsymbol{x}_t, c_t^1, \boldsymbol{z}_t) &= \max_{\boldsymbol{a}_t \in \mathbb{A}_t(\boldsymbol{x}_t, c_t^1)} W_t(\boldsymbol{a}_t, \boldsymbol{x}_t, c_t^1, \boldsymbol{z}_t) \\
&\leq \max_{\boldsymbol{a}_t \in \mathbb{A}_t(\boldsymbol{x}_t, c_t^1)} W_t(\boldsymbol{a}_t, \boldsymbol{x}_t, c_t^2, \boldsymbol{z}_t) \\
&\leq \max_{\boldsymbol{a}_t \in \mathbb{A}_t(\boldsymbol{x}_t, c_t^2)} W_t(\boldsymbol{a}_t, \boldsymbol{x}_t, c_t^2, \boldsymbol{z}_t) \\
&= V_t(\boldsymbol{x}_t, c_t^2, \boldsymbol{z}_t).
\end{aligned}$$

The second inequality here follows from our assumption that the set of allowed final asset positions $\mathbb{H}_t$ is nondecreasing in c_t: this implies that $\mathbb{A}_t(\boldsymbol{x}_t, c_t^1) \subseteq \mathbb{A}_t(\boldsymbol{x}_t, c_t^2)$.

For concavity: For any positions $(\boldsymbol{x}_t^1, c_t^1)$ and $(\boldsymbol{x}_t^2, c_t^2)$ and trading strategies $\boldsymbol{a}_t^1$ and $\boldsymbol{a}_t^2$, let $(\boldsymbol{x}_t^\alpha, c_t^\alpha) = \alpha(\boldsymbol{x}_t^1, c_t^1) + (1-\alpha)(\boldsymbol{x}_t^2, c_t^2)$ and let $\boldsymbol{a}_t^\alpha = \alpha\boldsymbol{a}_t^1 + (1-\alpha)\boldsymbol{a}_t^2$. Then we have

$$\begin{aligned}
W_t(\boldsymbol{a}_t^\alpha, \boldsymbol{x}_t^\alpha, c_t^\alpha, \boldsymbol{z}_t) &= \mathbb{E}\left[V_{t+1}(\tilde{\boldsymbol{r}}_{t+1} \cdot (\boldsymbol{x}_t^\alpha + \boldsymbol{a}_t^\alpha),\ r_f(c_t^\alpha - \mathbf{1}'\boldsymbol{a}_t^\alpha - \kappa(\boldsymbol{a}_t^\alpha)),\ \tilde{\boldsymbol{z}}_{t+1}) \mid \boldsymbol{z}_t\right] \\
&\geq \mathbb{E}\left[V_{t+1}\big(\tilde{\boldsymbol{r}}_{t+1} \cdot (\boldsymbol{x}_t^\alpha + \boldsymbol{a}_t^\alpha),\ r_f(c_t^\alpha - \mathbf{1}'\boldsymbol{a}_t^\alpha - \alpha\kappa(\boldsymbol{a}_t^1) - (1-\alpha)\kappa(\boldsymbol{a}_t^2)),\ \tilde{\boldsymbol{z}}_{t+1}\big) \mid \boldsymbol{z}_t\right] \\
&\geq \alpha\mathbb{E}\left[V_{t+1}\big(\tilde{\boldsymbol{r}}_{t+1} \cdot (\boldsymbol{x}_t^1 + \boldsymbol{a}_t^1),\ r_f(c_t^1 - \mathbf{1}'\boldsymbol{a}_t^1 - \kappa(\boldsymbol{a}_t^1)),\ \tilde{\boldsymbol{z}}_{t+1}\big) \mid \boldsymbol{z}_t\right] \\
&\quad +(1-\alpha)\mathbb{E}\left[V_{t+1}\big(\tilde{\boldsymbol{r}}_{t+1} \cdot (\boldsymbol{x}_t^2 + \boldsymbol{a}_t^2),\ r_f(c_t^2 - \mathbf{1}'\boldsymbol{a}_t^2 - \kappa(\boldsymbol{a}_t^2)),\ \tilde{\boldsymbol{z}}_{t+1}\big) \mid \boldsymbol{z}_t\right] \\
&= \alpha W_t(\boldsymbol{a}_t^1, \boldsymbol{x}_t^1, c_t^1, \boldsymbol{z}_t) + (1-\alpha) W_t(\boldsymbol{a}_t^2, \boldsymbol{x}_t^2, c_t^2, \boldsymbol{z}_t).
\end{aligned}$$

The first inequality follows by using the assumption that the transaction cost function $\kappa(\boldsymbol{a}_t)$ is convex in $\boldsymbol{a}_t$ and then using the nondecreasing in c_t part of the induction hypothesis. The next inequality follows from the concavity part of the induction hypothesis.

Suppose $\boldsymbol{a}_t^{1*}$ and $\boldsymbol{a}_t^{2*}$ are optimal trades given positions $(\boldsymbol{x}_t^1, c_t^1)$ and $(\boldsymbol{x}_t^2, c_t^2)$ and let $\boldsymbol{a}_t^{\alpha*} = \alpha\boldsymbol{a}_t^{1*} + (1-\alpha)\boldsymbol{a}_t^{2*}$. The concavity result above then implies

$$\begin{aligned}
V_t(\boldsymbol{x}_t^\alpha, c_t^\alpha, \boldsymbol{z}_t) &= \max_{\boldsymbol{a}_t \in \mathbb{A}_t(\boldsymbol{x}_t^\alpha, c_t^\alpha)} W_t(\boldsymbol{a}_t, \boldsymbol{x}_t^\alpha, c_t^\alpha, \boldsymbol{z}_t) \\
&\geq W_t(\boldsymbol{a}_t^{\alpha*}, \boldsymbol{x}_t^\alpha, c_t^\alpha, \boldsymbol{z}_t) \\
&\geq \alpha W_t(\boldsymbol{a}_t^{1*}, \boldsymbol{x}_t^1, c_t^1, \boldsymbol{z}_t) + (1-\alpha) W_t(\boldsymbol{a}_t^{2*}, \boldsymbol{x}_t^2, c_t^2, \boldsymbol{z}_t) \\
&= \alpha V_t(\boldsymbol{x}_t^1, c_t^1, \boldsymbol{z}_t) + (1-\alpha) V_t(\boldsymbol{x}_t^2, c_t^2, \boldsymbol{z}_t)
\end{aligned}$$

The first inequality above follows from our assumption that the set of allowed final asset positions $\mathbb{H}_t$ is convex: this implies that $\boldsymbol{a}_t^{\alpha*}$ is in $\mathbb{A}_t(\boldsymbol{x}_t^\alpha, c_t^\alpha)$ and is thus feasible but not necessarily optimal for the optimization problem in the first line above. □

Proof of Proposition 2.2. Proofs for the first two parts of this proposition can be constructed in much the same way as the proof of Proposition 2.1 given above.

Part 3: Given asset position $(\boldsymbol{x}_t, c_t)$ and market state $\boldsymbol{z}_t$ consider a trading strategy starting in this state that is optimal with transaction costs and achieves the value $V_t(\boldsymbol{x}_t, c_t, \boldsymbol{z}_t)$. This trading strategy is feasible for the model without transaction costs (because the sets of possible asset positions is assumed to be nondecreasing in cash) and, without transaction costs, would yield an expected utility μ that is at least as large as $V_t(\boldsymbol{x}_t, c_t, \boldsymbol{z}_t)$. Since this strategy is feasible but not necessarily optimal in the model without transaction costs, we know $V_t^f(\mathbf{1}'\boldsymbol{x}_t + c_t, \boldsymbol{z}_t) \geq \mu \geq V_t(\boldsymbol{x}_t, c_t, \boldsymbol{z}_t)$. □

Proof of Proposition 4.1. A proof of a more general version of this result may be found in Brown, Smith, and Sun (2009); a proof for this version of the result is given in the discussion following the Proposition. □

Derivation of Rolling-Buy-and-Hold Penalty. Assuming proportional transaction costs given by equation (1), the Taylor series expansion of (23) about $\boldsymbol{a}^*$ yields a generating function of the form

$$\begin{aligned} g_t(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z}) &= \mathbb{E}\left[V_{t+h}^f\Big((\tilde{\boldsymbol{r}}_{t+h} \cdot \ldots \cdot \tilde{\boldsymbol{r}}_{t+2})'\boldsymbol{x}_{t+1}(\boldsymbol{a}^*, \boldsymbol{r}) + r_f^{h-1} c_{t+1}(\boldsymbol{a}^*),\ \tilde{\boldsymbol{z}}_{t+h} \Big) \mid \boldsymbol{z}_{t+1} \right] \qquad (35) \\ &+ \sum_{\tau=1}^{t}\sum_{i=1}^{n} \mathbb{E}\left[V_{t+h}^{f\prime}(-)(\tilde{r}_{t+h,i} \ldots \tilde{r}_{t+2,i}) \mid \boldsymbol{z}_{t+1} \right] \left(\frac{\partial x_{t+1,i}}{\partial a_{\tau,i}^+} \left(a_{\tau,i}^+ - a_{\tau,i}^{*+}\right) + \frac{\partial x_{t+1,i}}{\partial a_{\tau,i}^-} \left(a_{\tau,i}^- - a_{\tau,i}^{*-}\right) \right) \\ &+ \sum_{\tau=1}^{t}\sum_{i=1}^{n} \mathbb{E}\left[V_{t+h}^{f\prime}(-) \mid \boldsymbol{z}_{t+1} \right] r_f^{h-1} \left(\frac{\partial c_{t+1,i}}{\partial a_{\tau,i}^+} \left(a_{\tau,i}^+ - a_{\tau,i}^{*+}\right) + \frac{\partial c_{t+1,i}}{\partial a_{\tau,i}^-} \left(a_{\tau,i}^- - a_{\tau,i}^{*-}\right) \right) \end{aligned}$$

where

$$\begin{aligned} V_{t+h}^{f\prime}(-) &= V_{t+h}^{f\prime}\Big((\tilde{\boldsymbol{r}}_{t+h} \cdot \ldots \cdot \tilde{\boldsymbol{r}}_{t+2})'\boldsymbol{x}_{t+1}(\boldsymbol{a}^*, \boldsymbol{r}) + r_f^{h-1} c_{t+1}(\boldsymbol{a}^*),\ \tilde{\boldsymbol{z}}_{t+h} \Big) \\ \frac{\partial x_{t+1,i}}{\partial a_{\tau,i}^+} &= \prod_{\tau'=\tau+1}^{t+1} r_{\tau',i} \\ \frac{\partial x_{t+1,i}}{\partial a_{\tau,i}^-} &= \prod_{\tau'=\tau+1}^{t+1} r_{\tau',i} \\ \frac{\partial c_{t+1,i}}{\partial a_{\tau,i}^+} &= -\, r_f^{t+1-\tau} \left(1 + \delta_i^+\right) \\ \frac{\partial c_{t+1,i}}{\partial a_{\tau,i}^-} &= -\, r_f^{t+1-\tau} \left(1 - \delta_i^-\right) \end{aligned}$$

As with the modified-one-step penalty, using a generating function of the form of (35) with the formula for a "good" penalty (21), we obtain a dual feasible penalty π that is linear in the positive and negative components of $\boldsymbol{a}$ for any sequence of returns $\boldsymbol{r}$ and market states $\boldsymbol{z}$. To calculate the "good" penalty, we need to evaluate the expectations (over returns $\boldsymbol{r}_{t+1}$ and the market state $\boldsymbol{z}_{t+1}$). The equations for the generating function (35) are more complicated than the generating functions for the modified one-step penalty (22) because (35) involves expectations of the frictionless value function and its derivatives over period t to $t+h$; these will be calculated using a quadrature scheme. This additional complication leads to these bounds being somewhat more time-consuming to compute than the modified one-step bounds; see Table 1 in §5.2. □

Proof of Proposition 4.2. Part (a): To streamline our notation, we define $f(\alpha, \boldsymbol{r}, \boldsymbol{z}) = U(\hat{w}_T(\alpha(\boldsymbol{r}, \boldsymbol{z}), \boldsymbol{r}))$ and $F(\alpha) = \mathbb{E}\left[f(\alpha, \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})\right] = \mathbb{E}\left[U(\hat{w}_T(\alpha(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}), \tilde{\boldsymbol{r}}))\right]$. Note that since $\hat{w}_T$ is a concave function and U is concave and nondecreasing, $f(\alpha, \boldsymbol{r}, \boldsymbol{z})$ is concave in α for each $(\boldsymbol{r}, \boldsymbol{z})$; this implies that $F(\alpha)$ is also concave in α. The (one-sided) directional derivatives of f and F at α in direction δ are defined as

$$\begin{aligned} f'(\alpha, \delta, \boldsymbol{r}, \boldsymbol{z}) &= \lim_{\epsilon \to 0^+} \frac{f(\alpha + \epsilon\delta, \boldsymbol{r}, \boldsymbol{z}) - f(\alpha, \boldsymbol{r}, \boldsymbol{z})}{\epsilon} \\ F'(\alpha, \delta) &= \lim_{\epsilon \to 0^+} \frac{F(\alpha + \epsilon\delta) - F(\alpha)}{\epsilon} \end{aligned}$$

The concavity of $f(\alpha, \boldsymbol{r}, \boldsymbol{z})$ and $F(\alpha)$ in α implies these directional derivatives exist (in the extended real numbers). If $U(w)$ is differentiable in w at $\hat{w}_T(\alpha(\boldsymbol{r}, \boldsymbol{z}), \boldsymbol{r})$ and $\hat{w}_T(\boldsymbol{a}, \boldsymbol{r})$ is differentiable in $\boldsymbol{a}$ at $(\alpha(\boldsymbol{r}, \boldsymbol{z}), \boldsymbol{r})$, then the gradient of $U \circ \hat{w}_T$ exists at this point and $f'(\alpha, \delta, \boldsymbol{r}, \boldsymbol{z}) = \nabla_{\boldsymbol{a}} U(\hat{w}_T(\alpha(\boldsymbol{r}, \boldsymbol{z}), \boldsymbol{r}))' \ \delta$.

Let $\mathcal{A}$ denote the set of feasible trading strategies for the real model and $\hat{\mathcal{A}}$ denote the set of feasible trading strategies for the modified model. Since $\mathbb{A}_t \subseteq \hat{\mathbb{A}}_t$, the set of trading strategies in the modified model is no smaller than that of the true model, i.e., $\mathcal{A} \subseteq \hat{\mathcal{A}}$. Moreover, the sets $\hat{\mathbb{A}}_t$ are convex, so the set of feasible trading strategies $\hat{\mathcal{A}}$ is also convex.

Given the concavity of F and convexity of $\hat{\mathcal{A}}$, a necessary and sufficient condition for $\hat{\alpha}^*$ to maximize $F(\alpha)$ over $\hat{\mathcal{A}}$ is that the directional derivatives of F at $\hat{\alpha}^*$ are nonpositive for all feasible directions:

$$F'(\hat{\alpha}^*, \alpha - \hat{\alpha}^*) \leq 0 \tag{36}$$

for all α in $\hat{\mathcal{A}}$; see, e.g., Friedlen and Nashed (1968).[1] A result of Bertsekas (1973; Prop 2.1)[2] says that

$$F'(\alpha, \delta) \quad = \quad \mathbb{E}\left[f'(\alpha, \delta, \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})\right] \ . \tag{37}$$

Bertsekas's result relies on the fact that $f(\alpha, \boldsymbol{r}, \boldsymbol{z})$ is concave in α for each $(\boldsymbol{r}, \boldsymbol{z})$ and assumes $f(\alpha, \boldsymbol{r}, \boldsymbol{z})$ is integrable for each α (i.e., $\mathbb{E}\left[|f(\alpha, \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})|\right] < \infty$), but does not require any other differentiability or integrability assumptions. (Bertsekas assumes that α lies in $\mathbb{R}^n$ but his proof also goes through when α lies in an infinite-dimensional convex set.) Combining (36) and (37) and using the fact that $\mathcal{A} \subseteq \hat{\mathcal{A}}$, we have

$$\mathbb{E}\left[f'(\hat{\alpha}^*, \alpha - \hat{\alpha}^*, \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})\right] \leq 0 \tag{38}$$

for all α in $\mathcal{A}$.

If $U \circ \hat{w}_T$ is differentiable, then (38) is equivalent to

$$\mathbb{E}\left[\hat{\pi}(\alpha(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}), \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})\right] = \mathbb{E}\left[\nabla_{\boldsymbol{a}} U(\hat{w}_T(\hat{\alpha}^*(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}), \tilde{\boldsymbol{r}}))' \ (\alpha(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}) - \hat{\alpha}^*(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}))\right] \leq 0 \tag{39}$$

for all α in $\mathcal{A}$. Thus, the gradient-based penalty $\hat{\pi}$ given by equation (29) is dual feasible.

If $U \circ \hat{w}_T$ is not differentiable, we can define the penalty $\hat{\pi}$ directly in terms of the directional derivative,

$$\hat{\pi}(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z}) \quad = \quad \lim_{\epsilon \to 0^+} \frac{U(\hat{w}_T(\hat{\alpha}^*(\boldsymbol{r}, \boldsymbol{z}) + \epsilon(\boldsymbol{a} - \hat{\alpha}^*(\boldsymbol{r}, \boldsymbol{z})), \boldsymbol{r})) - U(\hat{w}_T(\hat{\alpha}^*(\boldsymbol{r}, \boldsymbol{z}), \boldsymbol{r}))}{\epsilon} \ . \tag{40}$$

The existence of these directional derivatives follows from concavity alone and, in this case, we have

$$\mathbb{E}\left[\hat{\pi}(\alpha(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}), \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})\right] = \mathbb{E}\left[f'(\hat{\alpha}^*, \alpha - \hat{\alpha}^*, \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})\right], \tag{41}$$

which is less than or equal to 0 by (38). Thus this $\hat{\pi}$ is also dual feasible.

Part (b): The proof for the case where $U(w)$ and $\hat{w}_T(\boldsymbol{a}, \boldsymbol{r})$ are differentiable is given in the text before the proposition, near equations (26) and (27). Here we show the result without assuming U and $\hat{w}_T$ are differentiable,

We first focus on a $(\boldsymbol{r}, \boldsymbol{z})$ particular scenario and for ease of notation, we will suppress the dependence on $(\boldsymbol{r}, \boldsymbol{z})$: we let $\alpha^* = \alpha^*(\boldsymbol{r}, \boldsymbol{z})$ and define the function $u : \mathbb{R}^{nT} \to \mathbb{R}^1$ as $u(\boldsymbol{a}) = U(w_T(\boldsymbol{a}, \boldsymbol{r}))$ and denote its directional derivative (in direction $\boldsymbol{d}$) by $u'(\boldsymbol{a}, \boldsymbol{d})$. The penalty π in this case is $\pi(\boldsymbol{a}) = u'(\alpha^*, \boldsymbol{a} - \alpha^*)$, and the inner problem is

$$\max_{\boldsymbol{a} \in \mathbb{A}(\boldsymbol{r})} \{u(\boldsymbol{a}) - u'(\alpha^*, \boldsymbol{a} - \alpha^*)\} \ .$$

Following standard results in convex analysis (e.g., Rockafellar 1970; Thm. 23.4),[3] we can express the

[1]Friedlen, D.M. and M.Z. Nashed, 1968. A note on one-sided directional derivatives, *Mathematics Magazine* **41**(3), 147-150.

[2]Bertsekas, D.P., 1973. Stochastic optimization Problems with nondifferentiable cost functionals, *Journal of Optimization Theory and Applications* **12**(2), 218-231.

[3]Rockafellar, R.T., 1970. Convex Analysis, Princeton University Press, Princeton, NJ.

directional derivative for the concave function u as

$$u'(\boldsymbol{a},\boldsymbol{d}) \quad = \quad \inf_{\boldsymbol{h}\in\partial u(\boldsymbol{a})} \boldsymbol{d}'\boldsymbol{h},$$

where $\partial u(\boldsymbol{a})$ is the superdifferential of u at $\boldsymbol{a}$, defined as the set of all supergradients at this point:

$$\partial u(\boldsymbol{a}) \quad = \quad \left\{\boldsymbol{h}\in\mathbb{R}^{nT} \ : \ u(\boldsymbol{a})+\boldsymbol{h}'(\boldsymbol{b}-\boldsymbol{a})\geq u(\boldsymbol{b}) \ \text{ for all } \ \boldsymbol{b}\in\mathbb{R}^{nT}\right\}.$$

We thus have:

$$\begin{aligned}
\max_{\boldsymbol{a}\in\mathbb{A}(\boldsymbol{r})}\{u(\boldsymbol{a})-u'(\alpha^*,\boldsymbol{a}-\alpha^*)\} &= \max_{\boldsymbol{a}\in\mathbb{A}(\boldsymbol{r})}\left\{u(\boldsymbol{a})-\inf_{\boldsymbol{h}\in\partial u(\alpha^*)}(\boldsymbol{a}-\alpha^*)'\boldsymbol{h}\right\}\\
&= \max_{\boldsymbol{a}\in\mathbb{A}(\boldsymbol{r})}\sup_{\boldsymbol{h}\in\partial u(\alpha^*)}\left\{u(\boldsymbol{a})+\boldsymbol{h}'(\alpha^*-\boldsymbol{a})\right\}\\
&= \sup_{\boldsymbol{h}\in\partial u(\alpha^*)}\left\{\boldsymbol{h}'\alpha^*+\max_{\boldsymbol{a}\in\mathbb{A}(\boldsymbol{r})}\left\{u(\boldsymbol{a})-\boldsymbol{h}'\boldsymbol{a}\right\}\right\}\\
&= \sup_{\boldsymbol{h}\in\partial u(\alpha^*)}\left\{\boldsymbol{h}'\alpha^*+u(\alpha^*)-\boldsymbol{h}'\alpha^*\right\}\\
&= u(\alpha^*)\ .
\end{aligned}$$

The fourth equality follows from the fact that for any $\boldsymbol{h}\in\partial u(\alpha^*)$, the definition of a supergradient implies

$$u(\boldsymbol{a})-\boldsymbol{h}'\boldsymbol{a} \quad \leq \quad u(\alpha^*)-\boldsymbol{h}'\alpha^*,$$

and since $\alpha^*=\alpha^*(\boldsymbol{r},\boldsymbol{z})\in\mathbb{A}(\boldsymbol{r})$, $\boldsymbol{a}=\alpha^*(\boldsymbol{r},\boldsymbol{z})$ is feasible and attains the maximum value.

Returning to the original notation, we have established that, for any $(\boldsymbol{r},\boldsymbol{z})$,

$$\max_{\boldsymbol{a}\in\mathbb{A}(\tilde{\boldsymbol{r}})}\{U(w_T(\boldsymbol{a},\boldsymbol{r}))-\pi(\boldsymbol{a},\boldsymbol{r},\boldsymbol{z})\}=U(w_T(\alpha^*(\boldsymbol{r},\boldsymbol{z}),\boldsymbol{r}))\ .$$

Taking expectations of this, we have:

$$\mathbb{E}\left[\max_{\boldsymbol{a}\in\mathbb{A}(\tilde{\boldsymbol{r}})}\{U(w_T(\boldsymbol{a},\tilde{\boldsymbol{r}}))-\pi(\boldsymbol{a},\tilde{\boldsymbol{r}},\tilde{\boldsymbol{z}})\}\right] \quad = \quad \mathbb{E}\left[U(w_T(\alpha^*(\tilde{\boldsymbol{r}},\tilde{\boldsymbol{z}}),\tilde{\boldsymbol{r}}))\right]\ .$$

Part(c): We have

$$\begin{aligned}
\max_{\boldsymbol{a}\in\mathbb{A}(\boldsymbol{r})}\{U(w_T(\boldsymbol{a},\boldsymbol{r}))-\hat{\pi}(\boldsymbol{a},\boldsymbol{r},\boldsymbol{z})\} &\leq \max_{\boldsymbol{a}\in\hat{\mathbb{A}}(\boldsymbol{r})}\{U(\hat{w}_T(\boldsymbol{a},\boldsymbol{r}))-\hat{\pi}(\boldsymbol{a},\boldsymbol{r},\boldsymbol{z})\}\\
&= U(\hat{w}_T(\hat{\alpha}^*(\boldsymbol{r},\boldsymbol{z}),\boldsymbol{r})).
\end{aligned}$$

The inequality follows from $\mathbb{A}(\boldsymbol{r})\subseteq\hat{\mathbb{A}}(\boldsymbol{r})$ and the condition $w_T\leq\hat{w}_T$. The next equality follows by arguments analogous to that part (b) that show the choice $\boldsymbol{a}=\hat{\alpha}^*(\boldsymbol{r},\boldsymbol{z})$ must attain the maximum and the observation that $\hat{\pi}(\hat{\alpha}^*(\boldsymbol{r},\boldsymbol{z}),\boldsymbol{r},\boldsymbol{z})=0$. □

A.2. Detailed Assumptions and Results for Numerical Experiments

Assumptions for model with three risky assets and predictability, from Lynch (2001):

Table A1: Data for Three-Asset Model with Predictability

	Large-cap	Mid-cap	Small-cap	Dividend Yield
Mean (a_r, a_z)	0.0053	0.0067	0.0072	0.0000
Regression coeff. (b_r, b_z)	0.0028	0.0049	0.0061	0.9700

Covariance (Σ_{ev})

	Large-cap	Mid-cap	Small-cap	Dividend Yield
Large-cap	0.002894	0.003532	0.003910	-0.000115
Mid-cap		0.004886	0.005712	-0.000144
Small-cap			0.007259	-0.000163
Dividend Yield				0.052900

Note Lynch (2001) considers four different models, each with three risky assets and a one-dimensional market state variable. His three risky assets correspond to portfolios of stocks that are either sorted by firm size (which we use) or by book-to-market ratio; the market state variable is either the term spread or a dividend yield (which we use). Lynch has normalized the dividend yield data so that it has unconditional mean equal to zero and unconditional standard deviation equal to one.

Assumptions for model with ten risky assets and no predictability:

Table A2: Data for the Ten-Asset Model without Predictability

	SP500	R1000V	RMidC	R2000V	MSCIW	NAREIT	LBUSGv	LBUSCp	LBMortBnd	USTreasBnd
Mean (a_r)	0.009850	0.010672	0.010868	0.011173	0.009073	0.009432	0.007239	0.008111	0.007828	0.006424

Covariance (Σ_e)

	SP500	R1000V	RMidC	R2000V	MSCIW	NAREIT	LBUSGv	LBUSCp	LBMortBnd	USTreasBnd
SP500	0.001887	0.001659	0.001897	0.001592	0.001531	0.000742	0.000112	0.000243	0.000194	0.000054
R1000V		0.001650	0.001721	0.001549	0.001339	0.000821	0.000109	0.000233	0.000183	0.000052
RMidC			0.002199	0.001995	0.001549	0.000968	0.000097	0.000242	0.000183	0.000039
R2000V				0.002194	0.001318	0.001166	0.000048	0.000189	0.000121	0.000011
MSCIW					0.001738	0.000634	0.000083	0.000188	0.000147	0.000041
NAREIT						0.001269	0.000110	0.000206	0.000142	0.000060
LBUSGv							0.000224	0.000259	0.000231	0.000128
LBUSCp								0.000346	0.000308	0.000150
LBMortBnd									0.000336	0.000144
USTreasBnd										0.000082

These parameters were estimated as the means and covariances of historical returns for these indices using monthly return data from 1981-2006. The indices are, from left to right, 5 stock indices: the S&P 500, the Russell 1000 Value Index, Russell MidCap Index, Russell 2000 Value, and MSCI World Gross index; Lehman Brothers' US government and corporate bond indices; Lehman Brothers' Fixed Rate Mortgage Backed Securities Index, a real estate index trust (NAREIT), and a composite index of 1-5 Year US Treasuries.

Numerical Results for Other Parameter Values:

Table A3: Results for the Three-Asset Model with Predictability

Parameters				Heuristic Strategies				Dual Bounds						Best Performance		
Horizon (T)	Risk Aversion Coeff. (γ)	Trans. Cost Rate (δ)		Cost Blind	One-Step	Modified One-Step	Rolling Buy-and-Hold	Zero Penalty	Modified One-Step	Rolling Buy-and-Hold	Frictionless Gradient Based	Modified Gradient	No Trans. Cost Bound	Best Strategy	Best Upper Bound	Gap
6	1.5	0.5%	CE Return (%)	3.97	5.37	5.92	5.92	55.01	6.22	6.43	6.51	6.06	7.09	5.92	6.06	0.14
			Mean Std. Error (%)	0.03	0.20	0.17	0.17	0.67	0.01	0.02	0.01	0.00		Rolling Buy-	Modified	0.17
			Turnover (%)	49.9	10.9	17.0	17.0	61.2	13.6	18.1	8.4	14.2	50.3	and-Hold	Gradient	
6	1.5	1.0%	CE Return (%)	0.94	1.89	4.99	4.99	50.53	5.32	5.31	6.03	5.14	7.09	4.99	5.14	0.15
			Mean Std. Error (%)	0.05	0.17	0.18	0.18	0.64	0.02	0.02	0.02	0.01		Modified	Modified	0.18
			Turnover (%)	49.5	3.4	15.2	15.2	48.3	11.8	13.5	7.0	13.5	50.3	One-Step	Gradient	
6	1.5	2.0%	CE Return (%)	-4.85	0.51	3.49	3.49	43.86	3.72	3.57	5.30	3.69	7.09	3.49	3.57	0.07
			Mean Std. Error (%)	0.09	0.00	0.21	0.20	0.60	0.02	0.01	0.04	0.01		Modified	Rolling Buy-	0.21
			Turnover (%)	48.6	0.0	11.4	11.2	37.0	9.9	10.9	5.3	12.0	50.3	One-Step	and-Hold	
6	3	0.5%	CE Return (%)	2.22	2.92	3.22	3.22	51.14	3.43	3.64	3.53	3.30	3.86	3.22	3.30	0.08
			Mean Std. Error (%)	0.01	0.12	0.10	0.10	0.65	0.01	0.02	0.01	0.00		Rolling Buy-	Modified	0.10
			Turnover (%)	26.8	5.5	9.5	9.4	61.2	8.5	14.1	4.9	10.6	26.9	and-Hold	Gradient	
6	3	1.0%	CE Return (%)	0.60	1.20	2.73	2.73	46.91	2.97	2.93	3.25	2.83	3.86	2.73	2.83	0.10
			Mean Std. Error (%)	0.03	0.08	0.13	0.13	0.62	0.01	0.01	0.01	0.00		Modified	Modified	0.13
			Turnover (%)	26.6	1.7	7.8	7.8	48.3	7.9	8.1	4.2	9.9	26.9	One-Step	Gradient	
6	3	2.0%	CE Return (%)	-2.55	0.51	1.95	1.95	40.56	2.18	2.03	2.81	2.11	3.86	1.95	2.03	0.08
			Mean Std. Error (%)	0.05	0.00	0.14	0.14	0.57	0.01	0.01	0.02	0.01		Rolling Buy-	Rolling Buy-	0.14
			Turnover (%)	26.4	0.0	5.7	5.6	37.0	6.7	6.0	3.3	8.8	26.9	and-Hold	and-Hold	
6	8	0.5%	CE Return (%)	1.13	1.40	1.49	1.50	41.13	1.63	1.78	1.63	1.55	1.75	1.50	1.55	0.05
			Mean Std. Error (%)	0.01	0.05	0.04	0.04	0.60	0.00	0.01	0.00	0.00		Rolling Buy-	Modified	0.04
			Turnover (%)	10.2	2.1	3.6	3.6	61.2	5.0	9.7	1.9	5.9	10.2	and-Hold	Gradient	
6	8	1.0%	CE Return (%)	0.51	0.76	1.31	1.31	37.62	1.49	1.42	1.52	1.38	1.75	1.31	1.38	0.06
			Mean Std. Error (%)	0.01	0.03	0.05	0.05	0.56	0.01	0.01	0.00	0.00		Rolling Buy-	Modified	0.05
			Turnover (%)	10.2	0.6	2.9	2.9	48.3	4.9	4.0	1.6	5.6	10.2	and-Hold	Gradient	
6	8	2.0%	CE Return (%)	-0.71	0.51	1.03	1.03	32.30	1.21	1.07	1.36	1.12	1.75	1.03	1.07	0.04
			Mean Std. Error (%)	0.02	0.00	0.06	0.05	0.49	0.01	0.00	0.01	0.00		Rolling Buy-	Rolling Buy-	0.06
			Turnover (%)	10.2	0.0	2.1	2.1	37.0	4.3	2.4	1.2	5.3	10.2	and-Hold	and-Hold	
															Average	0.09
															Minimum	0.04
															Maximum	0.15

Table A3: Results for the Three-Asset Model with Predictability

Parameters				Heuristic Strategies				Dual Bounds						Best Performance		
Horizon (T)	Risk Aversion Coeff. (γ)	Trans. Cost Rate (δ)		Cost Blind	One-Step	Modified One-Step	Rolling Buy-and-Hold	Zero Penalty	Modified One-Step	Rolling Buy-and-Hold	Frictionless Gradient Based	Modified Gradient	No Trans. Cost Bound	Best Strategy	Best Upper Bound	Gap
24	1.5	0.5%	CE Return (%)	5.18	6.45	6.87	6.87	53.78	7.16	7.71	7.26	7.15	7.40	6.87	7.15	0.27
			Mean Std. Error (%)	0.02	0.16	0.11	0.11	0.35	0.00	0.02	0.00	0.00		Modified	Modified	0.11
			Turnover (%)	36.8	3.4	6.1	5.8	90.6	4.6	16.1	2.3	3.5	37.6	One-Step	Gradient	
24	1.5	1.0%	CE Return (%)	3.01	3.84	6.56	6.57	49.33	6.91	6.99	7.13	6.90	7.40	6.57	6.90	0.33
			Mean Std. Error (%)	0.05	0.17	0.12	0.12	0.33	0.01	0.01	0.01	0.00		Rolling Buy-	Modified	0.12
			Turnover (%)	35.9	1.7	5.1	4.9	68.2	4.2	6.9	2.2	3.5	37.6	and-Hold	Gradient	
24	1.5	2.0%	CE Return (%)	-1.21	0.51	5.95	5.95	42.79	6.49	6.37	6.89	6.44	7.40	5.95	6.37	0.41
			Mean Std. Error (%)	0.09	0.00	0.16	0.16	0.31	0.01	0.01	0.01	0.01		Modified	Rolling Buy-	0.16
			Turnover (%)	34.3	0.0	3.9	3.8	48.4	3.8	4.3	2.0	3.4	37.6	One-Step	and-Hold	
24	3	0.5%	CE Return (%)	3.05	3.45	3.77	3.77	50.15	4.04	4.91	4.02	3.96	4.10	3.77	3.96	0.19
			Mean Std. Error (%)	0.01	0.10	0.05	0.05	0.37	0.00	0.02	0.00	0.00		Rolling Buy-	Modified	0.05
			Turnover (%)	17.5	1.8	4.7	4.4	90.6	3.7	17.3	1.4	2.9	17.7	and-Hold	Gradient	
24	3	1.0%	CE Return (%)	2.00	2.15	3.53	3.54	45.89	3.97	4.16	3.94	3.82	4.10	3.54	3.82	0.28
			Mean Std. Error (%)	0.02	0.08	0.08	0.08	0.36	0.01	0.01	0.00	0.00		Rolling Buy-	Modified	0.08
			Turnover (%)	17.3	0.8	3.4	3.2	68.2	3.5	7.0	1.3	2.9	17.7	and-Hold	Gradient	
24	3	2.0%	CE Return (%)	-0.07	0.51	3.18	3.18	39.59	3.85	3.77	3.79	3.57	4.10	3.18	3.57	0.39
			Mean Std. Error (%)	0.04	0.00	0.11	0.11	0.33	0.01	0.01	0.01	0.00		Rolling Buy-	Modified	0.11
			Turnover (%)	16.9	0.0	2.2	2.1	48.4	3.3	4.2	1.2	2.8	17.7	and-Hold	Gradient	
24	8	0.5%	CE Return (%)	1.43	1.59	1.70	1.71	40.99	1.87	2.83	1.80	1.78	1.83	1.71	1.78	0.08
			Mean Std. Error (%)	0.00	0.04	0.02	0.02	0.51	0.00	0.02	0.00	0.00		Rolling Buy-	Modified	0.02
			Turnover (%)	6.6	0.7	1.9	1.7	90.6	2.8	17.5	0.5	1.7	6.7	and-Hold	Gradient	
24	8	1.0%	CE Return (%)	1.03	1.12	1.61	1.62	37.27	1.89	2.13	1.77	1.74	1.83	1.62	1.74	0.12
			Mean Std. Error (%)	0.01	0.03	0.03	0.03	0.48	0.01	0.01	0.00	0.00		Rolling Buy-	Modified	0.03
			Turnover (%)	6.6	0.3	1.3	1.3	68.2	2.6	6.9	0.5	1.7	6.7	and-Hold	Gradient	
24	8	2.0%	CE Return (%)	0.22	0.51	1.48	1.48	31.73	1.98	1.94	1.72	1.65	1.83	1.48	1.65	0.17
			Mean Std. Error (%)	0.02	0.00	0.04	0.04	0.43	0.02	0.01	0.00	0.00		Rolling Buy-	Modified	0.04
			Turnover (%)	6.6	0.0	0.9	0.8	48.4	2.6	3.6	0.5	1.7	6.7	and-Hold	Gradient	
															Average	0.25
															Minimum	0.08
															Maximum	0.41

Table A3: Results for the Three-Asset Model with Predictability

Parameters				Heuristic Strategies				Dual Bounds						Best Performance		
Horizon (T)	Risk Aversion Coeff. (γ)	Trans. Cost Rate (δ)		Cost Blind	One-Step	Modified One-Step	Rolling Buy-and-Hold	Zero Penalty	Modified One-Step	Rolling Buy-and-Hold	Frictionless Gradient Based	Modified Gradient	No Trans. Cost Bound	Best Strategy	Best Upper Bound	Gap
48	1.5	0.5%	CE Return (%)	5.57	6.50	7.06	7.04	54.04	7.35	8.02	7.42	7.36	7.48	7.06	7.35	0.30
			Mean Std. Error (%)	0.02	0.12	0.07	0.07	0.25	0.00	0.01	0.00	0.00		Modified	Modified	0.07
			Turnover (%)	34.6	2.0	4.7	4.3	160.5	3.6	17.5	1.2	1.8	36.0	One-Step	One-Step	
48	1.5	1.0%	CE Return (%)	3.68	4.50	6.84	6.84	49.52	7.21	7.34	7.35	7.24	7.48	6.84	7.21	0.37
			Mean Std. Error (%)	0.04	0.14	0.08	0.08	0.24	0.01	0.01	0.00	0.00		Modified	Modified	0.08
			Turnover (%)	33.2	1.1	3.6	3.3	116.5	2.8	6.5	1.1	1.8	36.0	One-Step	One-Step	
48	1.5	2.0%	CE Return (%)	-0.01	0.51	6.35	6.33	42.85	7.04	6.97	7.23	7.01	7.48	6.35	6.97	0.62
			Mean Std. Error (%)	0.07	0.00	0.11	0.11	0.22	0.01	0.01	0.01	0.01		Modified	Rolling Buy-	0.11
			Turnover (%)	30.7	0.0	2.5	2.3	76.9	2.4	2.7	1.1	1.8	36.0	One-Step	and-Hold	
48	3	0.5%	CE Return (%)	3.23	3.41	3.85	3.85	50.38	4.19	5.27	4.11	4.08	4.15	3.85	4.08	0.24
			Mean Std. Error (%)	0.01	0.09	0.03	0.03	0.30	0.00	0.02	0.00	0.00		Rolling Buy-	Modified	0.03
			Turnover (%)	16.1	1.2	4.1	3.7	160.5	3.1	20.8	0.7	1.6	16.5	and-Hold	Gradient	
48	3	1.0%	CE Return (%)	2.31	2.42	3.65	3.66	46.07	4.23	4.44	4.07	4.02	4.15	3.66	4.02	0.35
			Mean Std. Error (%)	0.02	0.08	0.05	0.05	0.28	0.01	0.01	0.00	0.00		Rolling Buy-	Modified	0.05
			Turnover (%)	15.8	0.5	2.8	2.5	116.5	2.8	7.4	0.7	1.6	16.5	and-Hold	Gradient	
48	3	2.0%	CE Return (%)	0.49	0.51	3.32	3.32	39.65	4.28	4.22	3.99	3.89	4.15	3.32	3.89	0.56
			Mean Std. Error (%)	0.03	0.00	0.08	0.08	0.26	0.01	0.01	0.01	0.01		Modified	Modified	0.08
			Turnover (%)	15.2	0.0	1.7	1.5	76.9	2.6	3.1	0.7	1.5	16.5	One-Step	Gradient	
48	8	0.5%	CE Return (%)	1.49	1.56	1.73	1.73	41.58	1.96	3.33	1.83	1.82	1.84	1.73	1.82	0.09
			Mean Std. Error (%)	0.00	0.04	0.01	0.01	0.50	0.01	0.03	0.00	0.00		Rolling Buy-	Modified	0.01
			Turnover (%)	5.9	0.5	1.6	1.4	152.8	3.0	20.9	0.3	0.9	6.0	and-Hold	Gradient	
48	8	1.0%	CE Return (%)	1.14	1.20	1.65	1.66	37.94	2.07	2.39	1.81	1.80	1.84	1.66	1.80	0.14
			Mean Std. Error (%)	0.01	0.03	0.02	0.02	0.45	0.01	0.01	0.00	0.00		Rolling Buy-	Modified	0.02
			Turnover (%)	5.9	0.2	1.1	1.0	113.1	2.7	7.9	0.3	0.9	6.0	and-Hold	Gradient	
48	8	2.0%	CE Return (%)	0.42	0.51	1.53	1.53	32.33	2.28	2.26	1.78	1.76	1.84	1.53	1.76	0.23
			Mean Std. Error (%)	0.01	0.00	0.04	0.04	0.40	0.02	0.01	0.00	0.00		Modified	Modified	0.04
			Turnover (%)	5.8	0.0	0.7	0.6	76.0	2.8	3.4	0.3	0.9	6.0	One-Step	Gradient	
															Average	0.32
															Minimum	0.09
															Maximum	0.62

Table A4: Results for the Ten-Asset Model without Predictability

Parameters				Heuristic Strategies				Dual Bounds						Best Performance		
Horizon (T)	Risk Aversion Coeff. (γ)	Trans. Cost Rate (δ)		Cost Blind	One-Step	Modified One-Step	Rolling Buy-and-Hold	Zero Penalty	Modified One-Step	Rolling Buy-and-Hold	Frictionless Gradient Based	Modified Gradient	No Trans. Cost Bound	Best Strategy	Best Upper Bound	Gap
6	1.5	0.5%	CE Return (%)	10.54	11.26	12.49	12.49	54.50	12.49	12.49	13.13	12.52	13.62	12.49	12.49	0.00
			Mean Std. Error (%)	0.29	0.06	0.00	0.00	0.44	0.00	0.00	0.01	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	17.1	12.8	16.6	16.6	105.7	16.6	16.6	6.3	15.1	17.2	and-Hold	and-Hold	
6	1.5	1.0%	CE Return (%)	9.42	5.91	11.38	11.38	46.90	11.38	11.38	12.79	11.44	13.62	11.38	11.38	0.00
			Mean Std. Error (%)	0.28	0.00	0.00	0.00	0.39	0.00	0.00	0.02	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	17.0	0.0	16.5	16.5	81.4	16.5	16.5	4.6	14.9	17.2	and-Hold	and-Hold	
6	1.5	2.0%	CE Return (%)	7.84	5.91	9.21	9.21	37.05	9.21	9.21	12.36	9.30	13.62	9.21	9.21	0.00
			Mean Std. Error (%)	0.20	0.00	0.01	0.00	0.33	0.00	0.00	0.04	0.01		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	16.8	0.0	16.3	16.3	47.4	16.3	16.3	2.7	14.6	17.2	and-Hold	and-Hold	
6	3	0.5%	CE Return (%)	10.72	8.49	10.79	10.79	52.99	10.81	10.80	11.51	10.83	11.91	10.79	10.80	0.01
			Mean Std. Error (%)	0.00	0.08	0.01	0.01	0.40	0.00	0.00	0.01	0.00		Modified	Rolling Buy-	0.01
			Turnover (%)	17.9	6.5	16.6	16.6	105.7	16.6	16.6	5.7	14.7	18.0	One-Step	and-Hold	
6	3	1.0%	CE Return (%)	9.54	5.91	9.69	9.70	45.62	9.72	9.71	11.20	9.76	11.91	9.70	9.71	0.01
			Mean Std. Error (%)	0.00	0.00	0.01	0.01	0.36	0.00	0.00	0.02	0.00		Rolling Buy-	Rolling Buy-	0.01
			Turnover (%)	17.8	0.0	16.5	16.5	81.4	16.5	16.5	4.6	14.6	18.0	and-Hold	and-Hold	
6	3	2.0%	CE Return (%)	7.24	5.91	7.60	7.60	35.99	7.78	7.70	10.75	7.81	11.91	7.60	7.70	0.09
			Mean Std. Error (%)	0.00	0.00	0.06	0.06	0.31	0.00	0.00	0.03	0.01		Rolling Buy-	Rolling Buy-	0.06
			Turnover (%)	17.6	0.0	11.8	11.7	47.4	11.0	11.9	3.2	14.5	18.0	and-Hold	and-Hold	
6	8	0.5%	CE Return (%)	8.56	6.83	8.62	8.63	48.56	8.68	8.65	9.46	8.67	9.74	8.63	8.65	0.02
			Mean Std. Error (%)	0.00	0.07	0.01	0.01	0.43	0.00	0.00	0.01	0.00		Rolling Buy-	Rolling Buy-	0.01
			Turnover (%)	18.0	2.5	16.6	16.6	105.7	16.5	16.6	4.3	14.7	18.1	and-Hold	and-Hold	
6	8	1.0%	CE Return (%)	7.39	5.91	7.58	7.58	41.88	7.75	7.63	9.23	7.68	9.74	7.58	7.63	0.05
			Mean Std. Error (%)	0.00	0.00	0.03	0.03	0.38	0.00	0.00	0.01	0.00		Rolling Buy-	Rolling Buy-	0.03
			Turnover (%)	17.9	0.0	13.4	13.3	81.4	10.3	7.4	3.8	14.6	18.1	and-Hold	and-Hold	
6	8	2.0%	CE Return (%)	5.12	5.91	6.45	6.45	32.97	6.71	6.58	8.87	6.70	9.74	6.45	6.58	0.12
			Mean Std. Error (%)	0.00	0.00	0.10	0.10	0.33	0.01	0.00	0.02	0.00		Rolling Buy-	Rolling Buy-	0.10
			Turnover (%)	17.8	0.0	4.4	4.4	47.4	9.0	3.3	3.0	14.8	18.1	and-Hold	and-Hold	
															Average	0.03
															Minimum	0.00
															Maximum	0.12

Table A4: Results for the Ten-Asset Model without Predictability

Parameters				Heuristic Strategies				Dual Bounds						Best Performance		
Horizon (T)	Risk Aversion Coeff. (γ)	Trans. Cost Rate (δ)		Cost Blind	One-Step	Modified One-Step	Rolling Buy-and-Hold	Zero Penalty	Modified One-Step	Rolling Buy-and-Hold	Frictionless Gradient Based	Modified Gradient	No Trans. Cost Bound	Best Strategy	Best Upper Bound	Gap
24	1.5	0.5%	CE Return (%)	12.40	12.04	13.34	13.34	53.25	13.34	13.34	13.50	13.36	13.62	13.34	13.34	0.00
			Mean Std. Error (%)	0.08	0.03	0.00	0.00	0.21	0.00	0.00	0.00	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	15.0	3.3	4.1	4.1	155.3	4.1	4.1	2.1	3.9	4.8	and-Hold	and-Hold	
24	1.5	1.0%	CE Return (%)	11.52	5.92	13.06	13.06	45.55	13.06	13.06	13.38	13.10	13.62	13.06	13.06	0.00
			Mean Std. Error (%)	0.08	0.00	0.00	0.00	0.18	0.00	0.00	0.00	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	14.9	0.0	4.1	4.1	110.7	4.1	4.1	2.0	3.9	4.8	and-Hold	and-Hold	
24	1.5	2.0%	CE Return (%)	10.39	5.91	12.50	12.51	36.00	12.50	12.50	13.18	12.58	13.62	12.51	12.50	0.00
			Mean Std. Error (%)	0.07	0.00	0.00	0.00	0.16	0.00	0.00	0.01	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	12.6	0.0	4.1	4.1	57.9	4.1	4.1	1.8	3.9	4.8	and-Hold	and-Hold	
24	3	0.5%	CE Return (%)	11.54	8.99	11.63	11.63	51.81	11.65	11.65	11.82	11.65	11.91	11.63	11.65	0.02
			Mean Std. Error (%)	0.00	0.05	0.01	0.01	0.19	0.00	0.00	0.00	0.00		Modified	Rolling Buy-	0.01
			Turnover (%)	5.8	1.7	4.1	4.1	155.3	4.2	4.1	1.8	4.0	5.9	One-Step	and-Hold	
24	3	1.0%	CE Return (%)	11.16	5.92	11.35	11.35	44.32	11.38	11.37	11.72	11.40	11.91	11.35	11.37	0.02
			Mean Std. Error (%)	0.00	0.00	0.01	0.01	0.17	0.00	0.00	0.00	0.00		Modified	Rolling Buy-	0.01
			Turnover (%)	5.8	0.0	4.1	4.1	110.7	4.1	4.1	1.7	3.9	5.9	One-Step	and-Hold	
24	3	2.0%	CE Return (%)	10.42	5.91	10.07	10.04	34.96	10.90	10.91	11.55	10.88	11.91	10.42	10.88	0.46
			Mean Std. Error (%)	0.00	0.00	0.03	0.03	0.15	0.00	0.00	0.01	0.00		Cost Blind	Modified	0.00
			Turnover (%)	5.7	0.0	3.1	3.0	57.9	4.1	4.1	1.6	3.9	5.9		Gradient	
24	8	0.5%	CE Return (%)	9.36	7.09	9.45	9.45	47.47	9.55	9.54	9.67	9.48	9.74	9.45	9.48	0.03
			Mean Std. Error (%)	0.00	0.04	0.01	0.01	0.29	0.00	0.00	0.00	0.00		Rolling Buy-	Modified	0.01
			Turnover (%)	5.9	0.6	4.2	4.1	155.3	4.5	4.4	1.3	4.1	6.0	and-Hold	Gradient	
24	8	1.0%	CE Return (%)	8.98	5.92	8.96	8.95	40.72	9.31	9.30	9.60	9.22	9.74	8.98	9.22	0.23
			Mean Std. Error (%)	0.00	0.00	0.02	0.02	0.25	0.00	0.00	0.00	0.00		Cost Blind	Modified	0.00
			Turnover (%)	5.9	0.0	3.5	3.4	110.7	4.2	4.1	1.3	4.0	6.0		Gradient	
24	8	2.0%	CE Return (%)	8.24	5.91	7.47	7.47	32.08	9.95	10.00	9.48	8.70	9.74	8.24	8.70	0.46
			Mean Std. Error (%)	0.00	0.00	0.06	0.06	0.20	0.02	0.02	0.01	0.00		Cost Blind	Modified	0.00
			Turnover (%)	5.8	0.0	1.2	1.2	57.9	4.3	4.3	1.3	4.0	6.0		Gradient	
															Average	0.13
															Minimum	0.00
															Maximum	0.46

Table A4: Results for the Ten-Asset Model without Predictability

Parameters				Heuristic Strategies				Dual Bounds						Best Performance		
Horizon (T)	Risk Aversion Coeff. (γ)	Trans. Cost Rate (δ)		Cost Blind	One-Step	Modified One-Step	Rolling Buy-and-Hold	Zero Penalty	Modified One-Step	Rolling Buy-and-Hold	Frictionless Gradient Based	Modified Gradient	No Trans. Cost Bound	Best Strategy	Best Upper Bound	Gap
48	1.5	0.5%	CE Return (%)	13.00	12.27	13.48	13.48	53.19	13.48	13.48	13.56	13.50	13.62	13.48	13.48	0.00
			Mean Std. Error (%)	0.04	0.02	0.00	0.00	0.15	0.00	0.00	0.00	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	9.6	1.7	2.1	2.1	262.5	2.1	2.1	1.2	2.0	2.9	and-Hold	and-Hold	
48	1.5	1.0%	CE Return (%)	12.54	5.92	13.34	13.34	45.45	13.34	13.34	13.51	13.37	13.62	13.34	13.34	0.00
			Mean Std. Error (%)	0.04	0.00	0.00	0.00	0.13	0.00	0.00	0.00	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	9.5	0.0	2.1	2.1	174.4	2.1	2.1	1.2	2.0	2.9	and-Hold	and-Hold	
48	1.5	2.0%	CE Return (%)	11.83	5.91	13.06	13.06	35.87	13.06	13.06	13.40	13.13	13.62	13.06	13.06	0.00
			Mean Std. Error (%)	0.03	0.00	0.00	0.00	0.11	0.00	0.00	0.00	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	8.4	0.0	2.0	2.0	82.2	2.0	2.0	1.1	2.0	2.9	and-Hold	and-Hold	
48	3	0.5%	CE Return (%)	11.67	9.19	11.76	11.76	51.77	11.80	11.80	11.87	11.79	11.91	11.76	11.79	0.03
			Mean Std. Error (%)	0.00	0.04	0.01	0.01	0.14	0.00	0.00	0.00	0.00		Modified	Modified	0.01
			Turnover (%)	4.0	0.9	2.1	2.1	262.5	2.4	2.4	1.0	2.0	4.1	One-Step	Gradient	
48	3	1.0%	CE Return (%)	11.44	5.92	11.62	11.62	44.25	11.66	11.65	11.82	11.67	11.91	11.62	11.65	0.03
			Mean Std. Error (%)	0.00	0.00	0.01	0.01	0.13	0.00	0.00	0.00	0.00		Modified	Rolling Buy-	0.01
			Turnover (%)	4.0	0.0	2.1	2.1	174.4	2.1	2.1	1.0	2.0	4.1	One-Step	and-Hold	
48	3	2.0%	CE Return (%)	10.96	5.91	10.55	10.51	34.85	11.48	11.50	11.73	11.42	11.91	10.96	11.42	0.46
			Mean Std. Error (%)	0.00	0.00	0.03	0.03	0.11	0.00	0.00	0.01	0.00		Cost Blind	Modified	0.00
			Turnover (%)	4.0	0.0	1.6	1.5	82.2	2.0	2.0	1.0	2.0	4.1		Gradient	
48	8	0.5%	CE Return (%)	9.49	7.17	9.57	9.58	47.24	9.76	9.76	9.70	9.61	9.74	9.58	9.61	0.03
			Mean Std. Error (%)	0.00	0.04	0.02	0.02	0.24	0.01	0.01	0.00	0.00		Rolling Buy-	Modified	0.02
			Turnover (%)	4.1	0.3	2.2	2.1	249.5	3.4	3.5	0.8	2.1	4.1	and-Hold	Gradient	
48	8	1.0%	CE Return (%)	9.25	5.92	9.22	9.19	40.69	9.64	9.64	9.67	9.49	9.74	9.25	9.49	0.24
			Mean Std. Error (%)	0.00	0.00	0.02	0.02	0.20	0.00	0.00	0.00	0.00		Cost Blind	Modified	0.00
			Turnover (%)	4.1	0.0	1.9	1.8	171.6	2.7	2.7	0.8	2.1	4.1		Gradient	
48	8	2.0%	CE Return (%)	8.77	5.91	7.63	7.62	31.95	12.02	12.29	9.61	9.24	9.74	8.77	9.24	0.47
			Mean Std. Error (%)	0.00	0.00	0.05	0.05	0.17	0.07	0.07	0.00	0.00		Cost Blind	Modified	0.00
			Turnover (%)	4.1	0.0	0.6	0.6	82.0	6.0	6.1	0.8	2.0	4.1		Gradient	
															Average	0.14
															Minimum	0.00
															Maximum	0.47