

Evaluating Income Streams: A Decision Analysis Approach

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Most important decision problems—virtually all capital investments and planning situations—involve risky cash flows with uncertainties that are resolved over time. In most of these problems, the decision-maker has access to financial markets and may borrow and lend to smooth consumption over time. Yet, because of the difficulty of incorporating these borrowing and lending decisions into the evaluation models, these opportunities are rarely explicitly modeled in decision and risk analyses of these investments. In this paper, we study the errors induced by failing to account for these borrowing and lending decisions, and we develop extensions to the standard decision and risk analysis procedures that, given certain market and preference assumptions, take these borrowing and lending opportunities into account without overburdening the evaluation models.

(Decision Analysis; Temporal Resolution of Uncertainty; Risk Profiles)

1. Introduction

When evaluating a risky project using decision and risk analysis techniques, analysts typically construct a model that calculates project cash flows and the net present value (NPV) of these cash flows given different settings of input variables. To examine the risks of the projects, we assign probability distributions to the input variables and calculate a probability distribution, called a *risk profile*, showing the likelihood of the different possible NPVs. To determine the value of the project and the optimal investment strategy, we assess a utility function on NPV and use it to determine certainty equivalents for each possible strategy. The decision-maker's time and risk preferences are thus modeled separately: the NPV function captures preferences among deterministic cash flow streams and the utility function describes the decision-maker's preferences for uncertain present value amounts.

In applying this methodology, analysts typically argue that NPVs should be calculated by discounting at the interest rate for risk-free borrowing and lending. The reason for this is that, given any deterministic cash flow stream, the decision-maker could borrow and lend

over time and convert this cash flow stream into its present value equivalent. Conversely, given any current dollar amount, by borrowing and lending, the decision-maker could obtain any cash flow stream yielding the same NPV.¹ Consequently, the decision-maker should be indifferent between any two deterministic cash flow streams with the same NPV. In this argument, the cash flows in the project model are interpreted as *income* the decision-maker receives but does not necessarily *consume*: given an income stream, the decision-maker will borrow and lend to optimally spread consumption over time according to his own preferences.

Unfortunately, if we take this income interpretation seriously, we find that the decision-maker's preferences for risky projects exhibit a sensitivity to the time at which uncertainties are resolved that cannot be captured using standard decision analysis procedures. For example, consider two investments that both have a 50-50 chance of paying zero or one million dollars in 10

¹ The same argument is sometimes made with the "opportunity cost of funds" instead of the risk-free rate; e.g., "since you can get a 12 percent return elsewhere, you should discount at 12 percent."

years: in the first case, a coin is tossed today and the decision-maker learns today whether he will receive one million dollars 10 years from now; and in the second case, the coin is tossed 10 years from now. Any decision-maker who can borrow and lend money would prefer the first investment to the second because he could use the information provided by the coin toss to better plan his consumption over time. If, for example, the decision-maker learns that he will receive one million dollars in 10 years, he could borrow against this future income and increase his consumption today (e.g., buy a bigger house, fancy car, etc.). Yet, because these two investments have identical distributions for income in each period, they would appear identical in any expected utility analysis where utilities are based on income streams alone: no matter how you calculate NPVs or what utility function you use, these two gambles would have identical risk profiles and identical expected utilities. Even first-order stochastic dominance fails, because you could subtract a small amount from the gamble with early resolution to yield a gamble that is dominated (in income terms) but yet preferred to the gamble with delayed resolution.

These problems associated with applying expected utility methods directly to income streams have long been known, having been recognized in Markowitz (1959, Chapters 10–11) and Matheson and Howard (1968). In the economics literature, the problem of evaluating income streams with delayed resolution is sometimes referred to as the “temporal risk problem” and has been studied in Mossin (1969), Dréze and Modigliani (1972), Spence and Zeckhauser (1972), Kreps and Porteus (1978, 1979), and Machina (1984).²

Even though the problems associated with evaluating income streams are well known, there does not yet appear to be a good solution. One solution appearing in the literature is to abandon the expected utility methods in favor of a non-expected utility approach, for example,

² In addition to these normatively oriented papers, there is a growing literature on descriptive attitudes toward the timing of resolution of uncertainty; see, for example, Chew and Ho (1994), Wu (1996a, b), and Albrecht and Weber (1997). In this literature, we find that psychological concerns such as hope and anxiety play a role and are traded off against the economic planning benefits provided by the early resolution of uncertainty.

by applying Kreps and Porteus’ (1978) recursive utility procedure or Chew and Epstein’s (1989) generalization of this procedure directly to income streams. While these procedures can capture a preference for early resolution of uncertainty, the assumptions of these procedures are generally not consistent with expected utility preferences for consumption (see Kreps and Porteus 1979) and therefore lack a strong normative basis. A second solution is to note that the problems with applying expected utility procedures to income streams are not problems of the procedures themselves, but rather the result of applying them at the wrong level. As LaValle (1992) suggests, if our fundamental preferences are for consumption, we should explicitly model the relevant borrowing and lending decisions and focus on consumption rather than income. Unfortunately, it is difficult to formulate, assess, and solve such “grand models.” Out of necessity, analysts typically build “small world” models that focus on the project without considering these borrowing and lending decisions. In practice then, the “solution” appears to be to ignore the problem.

This paper has two goals. The first goal is to study the errors associated with “ignoring the problem” and applying the standard decision and risk analysis procedures to income streams without fully modeling the background borrowing, lending, and consumption decisions. We define the value of the project as its “present certainty equivalent value”: the lump-sum amount received with certainty today that makes the decision-maker just indifferent to undertaking the project (as in Dréze and Modigliani 1972). We similarly define *effective NPVs* as amounts received today that yield a utility equal to that generated by the project in a particular scenario and define *effective risk profiles* as distributions of these effective NPVs. Comparing these “true values” to those generated by standard procedures, we show that the standard procedures implicitly assume uncertainties are resolved earlier than they are and, consequently, overestimate the true values of projects. Using Raiffa’s classic wildcatter problem (Raiffa 1968) as an example, we show that the errors in the standard procedures can be substantial.

The second goal of this paper is to describe some practical procedures for overcoming the difficulties associated with evaluating income streams. Though the

calculation of present certainty equivalent values, effective NPVs, and effective risk profiles is generally difficult, we can develop simple computational procedures if we are willing to assume that the decision-maker's preferences for consumption over time can be represented by an additive-exponential utility function. In this case, we can determine present certainty equivalent values using a variation on the standard decision tree "roll back" procedure and can calculate effective NPVs using a related formula. These procedures do not require explicit modeling of the background borrowing and lending decisions or a full assessment of the decision-maker's preferences for consumption over time, but yet generate the same values that would be found if the borrowing and lending decisions were explicitly modeled and a full utility function assessed.

The paper is organized as follows. In §2, we analyze Raiffa's "wildcatter" example using standard decision and risk analysis techniques; this example is used to illustrate subsequent definitions and results. In §3, we begin by formalizing the analytic framework of the paper and defining present certainty equivalent values. We then present the rollback procedure for calculating present certainty equivalent values in the additive exponential case and discuss properties of present certainty equivalent values. In §4, we define effective NPVs and effective risk profiles, describe computations in the additive exponential case, and study properties of these effective NPVs and effective risk profiles. Section 5 provides a graphical analysis of the relations between effective and actual NPVs and the present certainty equivalents. Section 6 provides a brief summary and conclusions. All proofs are given in an appendix.

2. An Illustrative Example

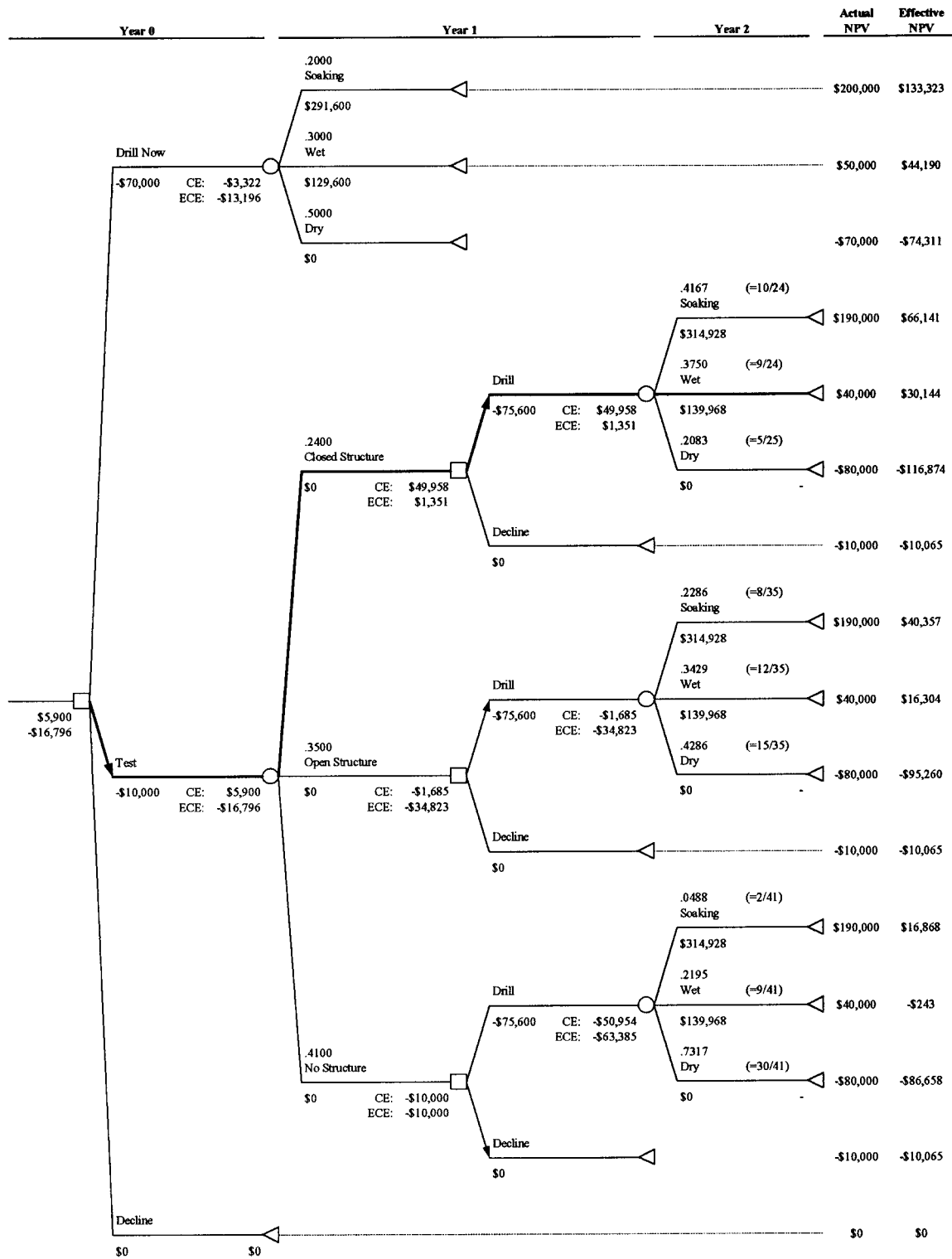
To illustrate the issues discussed in this paper, we reconsider Raiffa's classic wildcatter problem (Raiffa 1968), adding a temporal dimension to the problem as shown in the decision tree of Figure 1. The wildcatter has three alternatives: He can pay \$70,000 to drill a well now; he can pay \$10,000 to gather seismic information and delay the drilling decision one year, until after the seismic results are known; or he can decline to invest altogether. The wildcatter plans to sell the well upon completion, and one year after deciding to drill he will

receive a price for the well that depends on the amount of oil found. The probabilities of the various events are as given by Raiffa and are shown in the tree. The amounts shown beneath the branch labels in Figure 1 indicate the amount received when passing down that branch; these cash flow amounts were chosen so that their present values match the amounts used by Raiffa.

To compare our later results with those of standard practice, we first analyze the wildcatter problem using standard decision analysis procedures as described in, for example, McNamee and Celona (1990). In this approach, we calculate NPVs for each scenario by discounting cash flows at the risk-free rate and assign risk premiums using a utility function based on the decision-maker's preferences for gambles involving present value amounts. We assume a risk-free rate of 8 percent per year and assume that the wildcatter's risk preferences are captured by an exponential utility function with a risk tolerance (R) of \$200,000, so that the utility of a present value amount v is given by $-\exp(-v/R)$. This exponential form is commonly used in practice (see Howard 1988) and the risk tolerance of \$200,000 implies that the wildcatter is roughly indifferent between accepting or rejecting a gamble involving a 50-50 chance of winning \$200,000 and losing half that amount. Though it is often not made clear in the assessment process, we will assume (or assume that the decision-maker assumes) that the gambles used in assessing this utility function are resolved and paid immediately.

The endpoint values labeled actual NPVs are the NPVs of the cash flows along the path leading to that endpoint and are calculated using the risk-free rate of 8 percent. The upper rollback values in Figure 1 (labeled as CEs) are certainty equivalents calculated using this exponential utility function. The effective NPVs shown in the rightmost column and the lower rollback values (labeled ECEs) will be discussed later. Here we find that the optimal strategy is for the wildcatter to perform the test and, if the seismic test indicates the presence of a closed or open structure, drill the well. If the seismic test indicates no structure is present, then the wildcatter should not drill. The certainty equivalent for this strategy is \$5,900, as compared to certainty equivalents of $-\$3,322$ and $\$0$ for the "drill now" and "decline" strategies. The expected values are \$22,500 for the optimal

Figure 1 The Wildcatter Problem



testing strategy and \$20,000 for the “drill now” strategy, implying risk premiums equal to \$16,600 and \$23,322, respectively.

In addition to calculating certainty equivalents, analysts typically will describe the risks of a project by calculating a risk profile showing the distribution of possible NPVs generated by the project and will compare risk profiles for different possible strategies. Though the certainty equivalent summarizes the value of the project, decision-makers are typically interested in understanding the risks associated with the project as well as determining the project value. Cumulative risk profiles for the wildcatter problem are shown in Figure 2; these are labeled “actual risk profiles,” the effective risk profile will be discussed in §4. Here we see that, even though the testing strategy exposes the decision-maker to the greatest downside (the scenario where the wildcatter tests and still drills a dry well), the probability of the big loss associated with drilling a dry hole is greatly reduced compared to the strategy where the wildcatter drills now.

3. Present Certainty Equivalent Values

We begin in §3.1. by formalizing the analytic framework of the remainder of the paper. In §3.2. and 3.3. we define the present certainty equivalent value as a measure of the value of a project and discuss methods for computing these values. In §3.4. through 3.6., we discuss properties of present certainty equivalent values and compare them to the values generated using the standard decision and risk analysis procedures. The definition of the present certainty equivalent value (in §3.2.) and delay premiums (in §3.4.) are similar to those in Dréze and Modigliani (1972) who worked in a two-period setting rather than the multiperiod setting considered here. The rollback procedure for calculating present certainty equivalent values in the additive exponential case (in §3.3.) is a special case of the *integrated rollback procedure* developed in Smith and Nau (1995) (see also Smith 1996).

3.1. Basic Framework

We consider a single decision-maker who is deciding how to invest and consume over his lifetime. We assume that the decision-maker’s life (or planning hori-

zon) is divided into T time periods. At the end of each period, after that period’s uncertainties are resolved, the decision-maker decides how much to consume and how to invest the remainder of his assets. He can put his money in the bank and earn a risk-free return or invest in a variety of risky investments that generate uncertain streams of future income. The decisions and uncertainties are described by a decision tree (like that of Figure 1) where each node is “time stamped” to indicate the period when the uncertainty is resolved and the nodes are ordered according to their time of resolution. An *income stream* $(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_T)$ is a stochastic process describing the amounts received in each period along each path through the tree; the amount \tilde{x}_t is received in period t and the uncertainties about this amount are resolved over time as described in the tree.³

Consumption streams are similarly modeled as stochastic processes. We assume that the decision-maker makes decisions to maximize his expected utility of consumption where the utility for a (realized) consumption stream (c_0, c_1, \dots, c_T) is $U(c_0, c_1, \dots, c_T)$. To model the decision-maker’s borrowing and lending decisions we assume the availability of a risk-free bond with period- t price $(1 + r_f)^t$; the decision-maker adjusts his net borrowing and lending position by choosing the number of shares of the bond held. Letting β_t denote the number of risk-free bonds held from period t to period $t + 1$, given an investment generating an income stream (x_0, x_1, \dots, x_T) , the decision-maker consumes a net of $c_t = x_t + (\beta_{t-1} - \beta_t)(1 + r_f)^t$ in each period. Given a risky project $(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_T)$, the decision-maker chooses a borrowing and lending strategy $(\tilde{\beta}_0, \tilde{\beta}_1, \dots, \tilde{\beta}_{T-1})$ that solves⁴

$$\mathcal{U}(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_T) \equiv \max_{(\tilde{\beta}_0, \tilde{\beta}_1, \dots, \tilde{\beta}_{T-1})} E[U(\tilde{c}_0, \tilde{c}_1, \dots, \tilde{c}_T)]$$

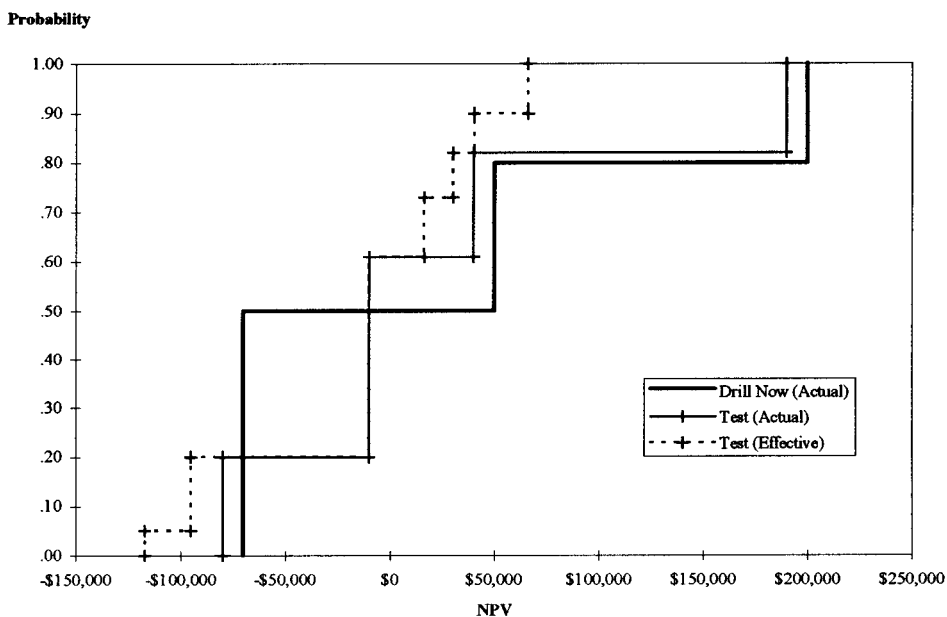
$$\text{subject to } \tilde{c}_t = \tilde{x}_t + (\tilde{\beta}_{t-1} - \tilde{\beta}_t)(1 + r_f)^t \text{ for all } t, \quad (1)$$

where $\mathcal{U}(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_T)$ denotes the utility derived from

³ We use tildes when referring to random variables or stochastic processes and omit them when referring to a particular realization of such a variable or process. The uncertainties can be modeled more formally as a “filtration” with income streams and consumption streams, etc. being stochastic processes adapted to this filtration.

⁴ The $\tilde{\beta}_t$ are stochastic because we don’t know the amounts of the risk-free security the decision-maker will choose to hold.

Figure 2 Risk Profiles for the Wildcatter Problem



a project $(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_T)$. The decision-maker's initial wealth is represented by β_{-1} , the number of shares of the risk-free security carried into the current period (period 0). The final β_T is taken to be zero so the decision-maker has no outstanding balances at the end of the horizon. We require the number of bonds held in period t (β_t) to be chosen based only on information available at time t . Otherwise, there are no restrictions on the amount the decision-maker may borrow, lend, or consume in any period. In particular, the decision-maker may choose negative β_t (corresponding to borrowing at the risk-free rate) and may consume negative amounts.

Given a choice among alternative investments, the decision-maker chooses among them according to the derived utility, $\mathcal{U}(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_T)$, selecting the one that leads to the highest expected utility when borrowing and lending opportunities are taken into account. We assume that the decision-maker's utility function is strictly increasing in each argument and strictly concave (so the decision-maker is risk averse with respect to the amount consumed in each period and is "multivariate risk averse" as well). To avoid technical difficulties, we also assume that for each project $(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_T)$ under consideration, there exists a borrowing and lending strategy that solves the optimization problem in (1).

3.2. Present Certainty Equivalent Values

We begin by defining what we mean by the value of a risky investment. We will define the value of a project $(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_T)$ as its *present certainty equivalent value* (or PCEV) which is the \hat{v}_0 such that

$$\mathcal{U}(\hat{v}_0, 0, \dots, 0) = \mathcal{U}(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_T). \quad (2)$$

In words, \hat{v}_0 is the lump-sum amount that makes the decision-maker just indifferent between receiving the lump sum (with certainty, immediately) and receiving the uncertain income stream over time; we assume the decision-maker adopts an optimal borrowing and lending strategy in both cases. The PCEV thus represents the breakeven selling price for a risky investment.⁵

For general utility functions, these PCEVs are difficult to compute. Even if the decision-maker's preferences for consumption are such that his preferences for current and future consumption streams are additive over time,

⁵ Though we focus on the breakeven selling price, we could obtain analogous results for the breakeven buying price of an investment instead. The breakeven buying price is given as the \hat{v}_0 such that $\mathcal{U}(0, 0, \dots, 0) = \mathcal{U}(\tilde{x}_0 - \hat{v}_0, \tilde{x}_1, \dots, \tilde{x}_T)$, i.e., the price (paid at time zero with certainty) that makes the decision-maker just indifferent about buying $(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_T)$.

the past, present, and future preferences are linked through the borrowing and lending decisions. For example, a decision-maker who consumes little in early periods and finds himself with substantial wealth in period t may be more willing to take risks in the future than he would if he had consumed more in earlier periods and consequently possesses little wealth in period t . In general, to find the value on the right side of (2), we must solve a nonlinear stochastic programming problem with decision variables corresponding to the choice of β_t in each period and each possible state of information in that period. Having found the solution to this problem, we determine \hat{v}_0 by solving the deterministic nonlinear programming problem on the left side of (2) with varying \hat{v}_0 to identify the value whose optimal utility matches the expected utility on the right side of (2).

3.3. The Additive-Exponential Case

While PCEVs are generally difficult to compute, given an additive-exponential utility function, these calculations are greatly simplified and we can determine PCEVs using a variation of the standard procedure for "rolling back" decision trees. In this case, we assume that the decision-maker's preferences for consumption can be represented by a utility function of the form:

$$U(c_0, c_1, \dots, c_T) = - \sum_{t=0}^T k_t \exp(-c_t/\rho_t). \quad (3)$$

Here the *utility weights* k_t and *risk tolerances* ρ_t are both assumed to be positive and can be thought of as capturing the decision-maker's time and risk preferences, respectively, for consumption. This additive exponential form is probably best thought of as an approximation to a decision-maker's true preferences for consumption and, as we argue in §6, its use can be viewed as analogous to the common practice of using an exponential utility function to capture a decision-maker's preferences for gambles resolved immediately. The computational procedures we present exploit both the exponential utility for consumption in each period and the additivity across periods. We discuss some sensitivity results that relax the exponential assumption in §3.6 below.

Given a decision-maker with an additive exponential utility function, we can determine PCEVs as follows.

First, we calculate NPVs for each endpoint in the tree by discounting all cash flows using the risk-free rate, as in the standard procedure. Then, we roll back the tree by choosing the maximum of the available alternatives at decision nodes, as usual. At chance nodes, we calculate *effective certainty equivalents*: given a chance node with successor values \tilde{v}_t , the rollback value, v_{t-1} , is equal to effective certainty equivalent ($ECE_t[\tilde{v}_t]$) given by using the exponential utility with *effective risk tolerance* R_t :

$$v_{t-1} = ECE_t[\tilde{v}_t] \equiv -R_t \ln(E[\exp(-\tilde{v}_t/R_t)]), \quad (4)$$

$$R_t \equiv \sum_{\tau=t}^T \frac{\rho_\tau}{(1+r_f)^\tau}. \quad (5)$$

We thus evaluate uncertainties resolved in different periods using different effective risk tolerances; the later the uncertainty is resolved the smaller the effective risk tolerance. The value given at the root of the tree is the project's PCEV. (Proof of this claim is given in the appendix.)⁶

The summing of period risk tolerances in Equation (5) reflects the decision-maker's ability to use the risk-free security to spread income risks over time. If, for example, the decision-maker were to receive a large windfall (say, a million dollars) in one period, he need not consume this income immediately. Instead he could increase his consumption in the current period and save the rest to finance increased consumption in subsequent periods. This spreading of risks over time is analogous to the risk sharing results or "theory of syndicates" developed in Wilson (1968). It is *as if* the decision-maker forms a syndicate with his "future selves" and optimally shares today's risks with his future selves by investing in the risk-free security. Wilson shows that groups of individuals with exponential utilities should share risks in proportion to their respective risk tolerances and should, as a Pareto efficient group, behave as

⁶ As a follow up to the previous footnote, there is no distinction between buying and selling prices in the case of the additive exponential utility function. This follows from the so called Δ -property for the exponential utility function: If you add a constant Δ in NPV to each endpoint in the tree, you increase each effective certainty equivalent, and ultimately the PCEV, by this constant Δ as well. Thus if we subtract the breakeven selling price (the PCEV) from each endpoint value, we obtain a project with 0 PCEV, so the breakeven buying price is equal to the breakeven selling price.

if they have an exponential utility with risk tolerance equal to the sum of the individual risk tolerances. Here the analogous results hold except the period risk tolerances are discounted to reflect the interest earned on the risk-free security.⁷

One perhaps surprising feature of this valuation procedure is that the PCEVs do not depend on the weights k_i in the utility function (3) or upon the decision-maker's initial wealth. Intuitively, the decision-maker adjusts his borrowing and lending strategy so as to bring his marginal preferences for consumption into alignment with the market rate for borrowing and lending. This optimal point depends on his utility weights and initial wealth, but once at this point, incremental cash flows are then valued at the market rate. The analogous results are true in the risk-sharing analogy: in the exponential case, the syndicate's evaluation criteria and amounts shared do not depend on the weights associated with each individual in the social welfare function. Changing the weights changes the deterministic payments from individual to individual (or how wealth is consumed over time), but efficient sharing always involves the same evaluation criteria and the same sharing of risky income across periods, regardless of the weights or the initial wealth.

We illustrate this procedure by calculating PCEVs for the wildcatter example. To make the example concrete, we will assume $T = 2$ and assume period risk tolerances (ρ_i) equal to \$71,858 for each period. Using a risk-free rate of 8 percent (as in the standard analysis of the previous section) and applying Equation (5), we find a period-0 effective risk tolerance R_0 of \$200,000. Thus, the decision-maker evaluates gambles resolved and paid immediately using an exponential utility function with a risk tolerance of \$200,000, as was assumed in the standard analysis. The NPVs for each scenario are the same

as in the standard analysis and are labeled actual NPVs in Figure 1. We then calculate expected certainty equivalents recursively through the tree using effective risk tolerances corresponding to the period where the uncertainty is resolved. For example following the bold path in Figure 1, at the amount of oil node we use an effective risk tolerance given by Equation (5) as $R_2 = \rho_2 / (1 + r_f)^2 = \$71,858 / (1.08)^2 = \$61,607$ and find an effective certainty equivalent of:

$$-R_2 \ln\left(\frac{10}{24} \exp(-\$190,000/R_2) + \frac{9}{24} \exp(-\$40,000/R_2) + \frac{5}{24} \exp(\$80,000/R_2)\right) = \$1,351.$$

Continuing through the tree, we use an effective risk tolerance (R_1) of \$128,142 to evaluate the period 1 risks and find that, if the wildcatter tests, he should drill the well only if the seismic test reveals a closed structure. The PCEV of this testing strategy is $-\$7,366$. The PCEV for the strategy that was optimal in the standard analysis of the previous section—test and drill unless the test indicates no structure is present—is $-\$16,796$, as shown in the lower rollback values in Figure 1.⁸ The PCEV for the “drill now” alternative is $-\$13,196$. The optimal strategy now is to decline initially, yielding a PCEV of \$0.

3.4. Delay Premiums

Comparing the results of this example to the results of the standard analysis of the previous section, we see that the PCEV for the “drill now” alternative is \$9,874 less than the certainty equivalent calculated in the standard procedure ($= -\$3,322 - (-\$13,196)$). Similarly, the PCEV for the testing strategy that was optimal in the standard analysis is \$22,696 less than the standard certainty equivalent ($= \$5,900 - (-\$16,796)$). It is easy to see why this is the case: In the standard procedure, we evaluated all risks using the current risk tolerance of \$200,000. When calculating the “true” present certainty equivalent value, we evaluated uncertainties resolved in periods 1 and 2 using effective risk tolerances of \$128,142 and \$61,607, respectively. The greater risk aversion in the later periods leads to PCEVs that are less

⁷ Those familiar with the theory of risk sharing (as developed in Wilson 1968, Pratt and Zeckhauser 1992, and elsewhere) might conjecture that similar results would hold for the period utility functions in the broader class of HARA (hyperbolic absolute risk aversion) utility functions. This is not true. In this context, not all risks are “on the table” to be shared because the decision-maker cannot share today in a risk that is not resolved until some point in the future. Pratt and Zeckhauser show that, with the presence of unshared background risks, you obtain “nice” sharing rules only if the syndicate members all have exponential utilities.

⁸ In what follows, to facilitate comparisons with the standard procedure, we will focus on the testing strategy that was optimal in the standard analysis, even though it is no longer optimal.

than the certainty equivalents given by the standard analysis.

The same result holds in general for all utility functions, not just the additive exponential utility of Equation (3). In the standard decision and risk analysis procedure, the cash flows are discounted first and evaluated using a utility function based on preferences for gambles resolved and paid immediately. If we assume the decision-maker takes into consideration the impact of this income on his future consumption when responding to the assessment questions, the assessed utility function $u(v)$ is given by $\mathcal{U}(v, 0, \dots, 0)$ where \mathcal{U} is the derived utility function for income streams specified in Equation (1). The standard procedure then calculates certainty equivalents as $u^{-1}(E[u(\text{NPV}(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_T))])$; we refer to this quantity as the *present value certainty equivalent* (or PVCE) reflecting its definition as the certainty equivalent for a gamble involving present values. If all of the project uncertainties are resolved immediately (in period 0) before any consumption decisions are made, this PVCE will be equal to the PCEV. If, however, some uncertainties are not resolved until after some consumption decisions made, the PVCE will overstate a project's true value because it assumes more information is available than actually is.

We can interpret the difference between the standard PVCE and the "true" PCEV as a *delay premium* representing the cost associated with the delayed resolution of uncertainty or, equivalently, the value of resolving all project uncertainties before making any consumption decisions. Using this, we can write the present certainty equivalent value of a project as:

$$\begin{aligned} \text{PCEV} = & (\text{Expected Value}) \\ & - (\text{Risk Premium}) - (\text{Delay Premium}), \quad (6) \end{aligned}$$

where the risk premium is equal to the difference between the expected value and the PVCE. For example in the wildcatter problem, the "drill now" alternative, Equation (6) becomes

$$\text{PCEV} = \$20,000 - \$23,322 - \$9,874 = -\$13,196.$$

Here we see that this alternative is unattractive because of its risks: even if the uncertainties were resolved immediately (to give a delay premium of zero), this alternative would still be unattractive. For the testing strat-

egy that was optimal in the original analysis (drill unless the test indicates no structure is present), Equation (6) becomes

$$\text{PCEV} = \$22,500 - \$16,600 - \$22,696 = -\$16,796.$$

In this case, it is the delay premium that makes this strategy unattractive: If we were able to resolve the uncertainty immediately, the strategy would be attractive. In contrast, receiving the risk premium (\$16,660) as a lump-sum bonus would not be sufficient to make this strategy attractive.

3.5. The Effects of Timing

In general, given the NPVs for a project, project values are independent of the timing of the actual receipt of the cash flows. This independence of the timing of receipt of cash flows is easiest to see in the additive exponential case: the rollback procedure starts with endpoint NPVs but makes no mention of the actual time at which the cash flows are received. The same result holds for other utility functions as well. Because the decision-maker can borrow and lend to convert one cash flow into any other cash flow stream with the same NPV (discounting at the risk-free rate), any two projects with income streams generating the same NPVs in each scenario will have the same set of feasible consumption streams and lead to identical optimal consumption streams.⁹ Thus, regardless of the decision-maker's preferences for consumption over time or the form of his utility function, given access to perfect markets for borrowing and lending, he should discount cash flows at the risk-free rate.

Though the actual timing of receipt of cash flows is irrelevant to project values, project values do depend on the time at which the uncertainty about the cash flows is resolved. For example, in the rollback procedure for the additive-exponential case, one uses different risk tolerances to evaluate uncertainties resolved in different periods. In this case and in general, this dependence of PCEVs on the time of resolution is captured entirely

⁹ To prove this more formally, note that the budget constraint associated with the optimization problem (1) (together with the requirement that $\beta_T = 0$) implies that, in each state, the NPV of the amounts consumed (c_t) and the amounts received as income (x_t) plus the decision-maker's initial wealth (w_0) be equal, i.e., $\sum_{t=0}^T c_t / (1 + r_f)^t = \sum_{t=0}^T x_t / (1 + r_f)^t + w_0$.

through the delay premium in (6); both the expected NPV and the risk premium are independent of the time of resolution. Moreover, the later an uncertainty is resolved, the less information available at the consumption decisions, the greater the delay premium and the less the PCEV.

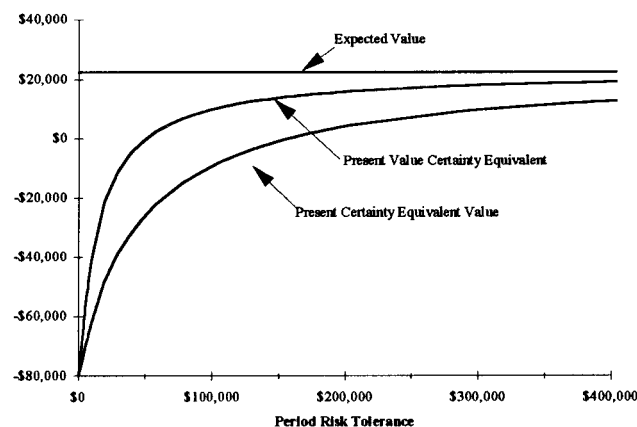
The decision-maker's ability to smooth income risks over times also affects the way we should think about the value of information. Suppose, for example, a clairvoyant could reveal the result of the seismic test in period 0 rather than in period 1, without changing the timing of the drilling decision or the sale of the well. In the additive-exponential case, this would mean that the seismic test uncertainty would be evaluated using the period-0 effective risk tolerance ($R_0 = \$200,000$) rather than the period-1 effective risk tolerance ($R_1 = \$128,142$); none of the drilling decisions or cash flows would be changed, but yet the PCEV would increase. In this case, the information does not change any of the project cash flows or projects decisions, yet the information increases the value of the project because of its impact on the background borrowing, lending, and consumption decisions. Thus, when we think about the decision analysis maxim that "Information has value if and only if it can change some decisions," we need to look beyond the project decisions that are modeled and consider the impact of information on the unmodeled borrowing, lending, and consumption decisions.

3.6. Sensitivity to Risk Attitude

In practice, a decision-maker's utility function will only be known approximately and it is important to understand the sensitivity of the PCEVs and the risk and delay premiums to changes in risk attitudes. In the additive exponential case, these sensitivity results are most transparent. In this case, we can see that if we were to simultaneously decrease each of the period risk tolerances (ρ_t), we would decrease each of the effective risk tolerances (R_t) and make the decision-maker more risk averse in each period. This would lead to smaller effective certainty equivalents and, working through the roll-back procedure, to reduced PCEVs. Decreasing each period risk tolerance similarly decreases the period-0 effective risk tolerance (R_0) and leads to a decrease in PVCEs and, therefore, an increase in risk premiums.

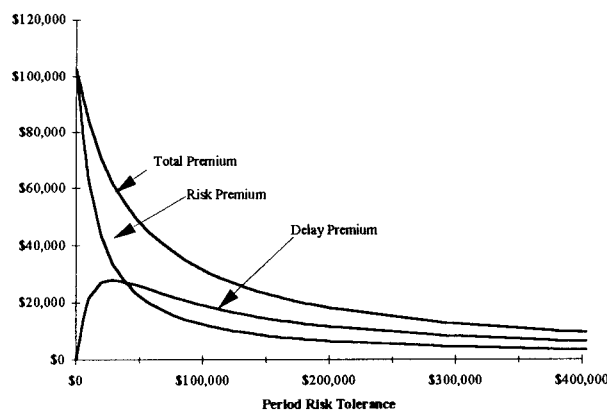
These sensitivity results are illustrated in Figures 3(a) and (b). In Figure 3(a), we show the PVCE, PCEV, and

Figure 3a Values as a Function of Risk Tolerance



expected value for the testing strategy that was optimal in the standard analysis of §2 (drill the well unless the seismic test indicates no structure is present) as a function of the period risk tolerances. We assume that all of the period risk tolerances are equal and vary them together; earlier we assumed $\rho_t = \$71,858$ for all t . In Figure 3(a) we see that as the period risk tolerances all increase, the PCEV and PVCE both approach the expected value of \$22,500. As the risk tolerances approach zero, the PCEV and PVCE both approach the worst possible outcome, $-\$80,000$, earned in the case where the wildcatter tests and still drills a dry hole. The associated risk and delay premiums are shown in Figure 3(b). Here we see that the risk premium, given by the difference between the expected value and the PVCE, is strictly decreasing in the period risk tolerances and

Figure 3b Premiums as a Function of Risk Tolerance



approaches 0 for large period risk tolerances. The delay premium, given by the difference between the PVCE and the PCEV, approaches zero for both large and small period risk tolerances. For small risk tolerances, risk aversion drives the value of a project towards the worst possible outcome and the delayed resolution of uncertainty causes no further penalty.

To generalize these sensitivity results beyond the additive-exponential case, we need to formalize what we mean by increasing or decreasing risk aversion in a more general setting. To do this, we will consider additive utility functions of the form

$$U(c_0, c_1, \dots, c_T) = \sum_{t=0}^T u_t(c_t),$$

where the $u_t(c_t)$ are increasing, concave functions describing the utility derived from consumption in period t . Here we say a period utility function u_t is "more risk averse" than another if it has a larger risk tolerance $\rho_t(c_t) = -u'_t(c_t)/u''_t(c_t)$ over the range of possible consumption levels; this is "global risk aversion" as in Pratt (1964). One overall utility function (U^A) is more risk averse than another (U^B) if each of U^A 's period utility functions is more risk averse than U^B 's corresponding period utility function. Here we find that if one utility function is more risk averse than another in this sense, it assigns lower PCEVs and PCVEs. (This is proven in the appendix.) This implies that, given an additive utility function, risk premiums increase with increasing risk aversion. In general, as in the exponential case, the delay premiums may either increase or decrease with increasing risk aversion, but since PCEVs decrease with increasing risk aversion, the sum of the risk and delay premiums must increase with increasing risk aversion.

One way to use this sensitivity result is to compute bounds on PCEVs for an additive, but nonexponential utility. If we can place upper and lower bounds ($\bar{\rho}_t$ and $\underline{\rho}_t$) and on the each period's risk tolerances $\rho_t(c_t) = -u'_t(c_t)/u''_t(c_t)$ (over the range of possible consumption levels), we can generate upper and lower bounds on the PCEV of a project using the rollback procedure for the additive-exponential case with upper and lower bounds on risk tolerances. The upper bound is given by taking period risk tolerances of ($\bar{\rho}_0, \bar{\rho}_1, \dots, \bar{\rho}_T$) in the

rollback procedure and the lower bound, similarly, using ($\underline{\rho}_0, \underline{\rho}_1, \dots, \underline{\rho}_T$) instead.

4. Effective NPVs and Effective Risk Profiles

As indicated in §2, in addition to calculating certainty equivalents, decision analysts typically present the decision-maker with a risk profile showing the distribution of possible NPVs generated by the project. Though the present certainty equivalent value summarizes the value of the project, decision-makers typically are interested in understanding the risks associated with the project as well as determining their value. Unfortunately, the distributions of NPVs (or risk profiles) generated by standard analyses do not properly describe the value of the project when the resolution of uncertainties is delayed. The timing of resolution of the uncertainties has no impact on the standard risk profile, and thus comparisons between risk profiles for projects with different timing can be misleading. As discussed in the introduction, even stochastic dominance comparisons between risk profiles fail because one project could be dominated by another and yet be preferred because its uncertainties are resolved earlier.

In this section, we introduce effective NPVs and effective risk profiles as tools for properly describing the risks associated with a project, taking into account the timing of resolution of uncertainties and the impact of this information on the background borrowing and lending decisions. These effective NPVs are analogous to PCEVs, but describe the value of a project given a particular resolution of uncertainties rather than the present value of a project before the uncertainties are resolved.

4.1. Effective NPVs and Effective Risk Profiles

The *effective NPV* of a project in a particular scenario is the amount such that the decision-maker is just indifferent between receiving this amount as a lump-sum immediately and having the income generated by the project with the uncertainties unfolding over time according to this scenario. More formally, let ($\bar{c}_0^*, \bar{c}_1^*, \dots, \bar{c}_T^*$) denote the optimal consumption stream corresponding to a particular project (i.e., the solution to the optimization problem (1)) and let (c_0^*, \dots, c_T^*) be the real-

ization of that consumption stream in a particular scenario. The effective NPV for this scenario is the amount v such that

$$\mathcal{U}(v, 0, \dots, 0) = U(c_0^*, c_1^*, \dots, c_T^*). \quad (7)$$

Thus the effective NPV is the present value amount that describes the utility derived from the project in a particular scenario, taking into account how the decision-maker adapts his borrowing and lending strategy in response to the resolution of uncertainties over time. An *effective risk profile* is the distribution of a project's effective NPVs where each effective NPV has the probability of the corresponding scenario. This effective risk profile can be interpreted as a "net present gamble" that, if resolved and paid immediately, would yield the same distribution of final utilities as the project. We refer to the standard NPVs as *actual NPVs*, because they refer to the NPV of the cash flows actually received over time. Similarly, the standard risk profile, showing the distribution of actual NPVs, is referred to as an *actual risk profile*.

4.2. The Additive-Exponential Case

Like PCEVs, effective NPVs and effective risk profiles are generally difficult to compute, except in the case where the decision-maker's utility function is an additive-exponential form. To describe the procedure for calculating effective NPVs in the additive-exponential case, we formalize the notion of "windfalls" mentioned briefly in §3.3. Let v_t denote the rollback value corresponding to a particular time- t state of information in the rollback procedure for calculating PCEVs described in §3.3. For $t > 0$, we define the *period- t windfall* as $w_t = v_t - v_{t-1}$ and, for $t = 0$, we define the windfall to be the PCEV, v_0 . These windfalls describe the change in project values as uncertainties are resolved over time. For example, following the bold path in Figure 1, we find that, if the wildcatter performs the seismic test, he receives a period-0 windfall of $-\$16,796$. If the seismic test reveals a closed structure in period 1, he receives windfall of $\$18,147$ ($=v_1 - v_0 = \$1,351 - (-\$16,796)$) as his valuation of the prospect would increase by this amount. If he were to subsequently drill a wet well, he would receive an additional windfall in period 2 of $\$38,649$ ($=v_2 - v_1 = \$40,000 - \$1,351$). These windfalls are denominated in period 0 terms and, by

construction, the sum of the windfalls along a path through a tree is equal to the corresponding scenario's actual NPV, which in this case is $\$40,000$.

In the case of the additive-exponential utility, we can characterize the decision-maker's consumption streams in terms of these windfalls in a very simple way and this leads to a closed form expression for the effective NPV in terms of the windfalls. Here we summarize the key steps in the derivation of this formula and refer the interested reader to the appendix for a detailed derivation. Let $(c_0^0, c_1^0, \dots, c_T^0)$ be the optimal consumption stream given 0 income in each period and let U^0 denote the utility generated by this consumption stream. Given a windfall stream (w_0, w_1, \dots, w_T) , the decision-maker consumes

$$c_t^* = c_t^0 + \rho_t \sum_{\tau=0}^t \frac{w_\tau}{R_\tau}, \quad (8)$$

in each period, so that each period's windfalls are spread over subsequent periods in proportion to their period risk tolerances. In the risk-sharing analogy of §3.3, each "future self" shares in today's risk in proportion to his risk tolerance. We can then show that the utility derived from this consumption stream can be written in terms of the windfall stream (w_0, w_1, \dots, w_T) and utility U^0 generated by c_t^0 as:

$$\frac{U^0}{R_0} \sum_{t=0}^T \left(\frac{\rho_t}{(1+r_f)^t} \exp\left(-\sum_{\tau=0}^t w_\tau/R_\tau\right) \right). \quad (9)$$

To find the effective NPV for a scenario, we equate the derived utility for period-0 income with utility received in that scenario (as specified in Equation (7)). From the rollback procedure of the previous section, given an additive-exponential utility, the derived utility for period-0 income, $u(v) = \mathcal{U}(v, 0, \dots, 0)$, is exponential with risk tolerance R_0 . We can fix the scale of this derived utility function by noting that zero income gives a utility U^0 , so that $u(v) = U^0 \exp(-v/R_0)$. The effective NPV v then equates the utility with immediate resolution $u(v)$ to the utility derived in that particular scenario $U(c_0^*, c_1^*, \dots, c_T^*)$. Solving this we find the effective NPV for a particular scenario can be expressed in terms of the corresponding windfall stream as:

$$v = -R_0 \ln \left(\sum_{t=0}^T \left(\frac{\rho_t}{(1+r_f)^t} \exp \left(- \sum_{\tau=0}^t w_\tau / R_\tau \right) \right) \right). \quad (10)$$

After some algebra, we can rewrite this expression as

$$v = -R_0 \ln \left(1 + \frac{1}{R_0} \sum_{t=0}^T R_t (\exp(-w_t/R_t) - 1) \cdot \exp \left(- \sum_{\tau=0}^{t-1} w_\tau / R_\tau \right) \right). \quad (11)$$

In this latter form, periods with zero windfalls drop out of the summation, which simplifies calculations when there are periods where consumption may occur but no project uncertainties are resolved. Note that in the additive-exponential case, these effective NPVs, like the PCEVs, are independent of the tradeoff weights (k_t) in the decision-maker's utility function and the decision-maker's initial wealth.

We can illustrate these calculations using the wildcatter example. Following the bold path in Figure 1, substituting the windfalls described earlier into Equation (10) or (11), we find an effective NPV of \$30,144 for this scenario. Thus the decision-maker is just indifferent between receiving \$30,144 in period 0 and receiving the project cash flows and having the uncertainties resolve over time following the bold path in Figure 1. The effective NPVs for the other scenarios are shown in Figure 1 and the effective risk profile for the testing strategy is shown in Figure 2.

4.3. Properties of Effective NPVs and Effective Risk Profiles

Comparing the effective NPVs for the wildcatter problem to the corresponding actual NPVs, we see that the effective NPV is less than its actual counterpart in every case. For example, in the scenario where the uncertainties resolve according to the bold path in Figure 3, the actual NPV exceeds the effective NPV by \$9,856 ($=\$40,000 - \$30,144$). The biggest difference occurs in the case where the seismic test reveals a closed structure and the wildcatter drills a "gushing" well; in this scenario, the difference is \$123,859 ($=\$190,000 - \$66,141$). This overestimation of project NPVs holds in general, not just for this example or for the exponential utility: If the decision-maker knew he was going to receive the

actual NPV in period 0, the actual and effective NPVs would be equal. Not knowing the actual NPV to be received, he cannot optimize his consumption for this scenario and, consequently, derives less utility than he would if he had this knowledge up front. Thus, just as the certainty equivalents generated using standard decision analysis procedures overestimate the true present value of a project, the standard NPVs overstate the value derived from a project in each scenario. Because the actual NPVs are greater than the effective NPVs in each scenario, the actual risk profiles stochastically dominate (in the first-order sense) the corresponding effective risk profiles. As in the case of the PCEVs, the difference between the actual and effective NPVs can be interpreted as a delay premium describing the loss in value in a particular scenario due to the delayed resolution of information. Looking at the effective risk profiles in Figure 2, we see that the errors are particularly large in the tails of the distribution.¹⁰

A perhaps surprising feature of these effective NPVs is that two scenarios with identical cash flows may have different effective NPVs.¹¹ For example in the wildcatter problem, consider the case where the wildcatter performs the seismic test and the well is "wet." Regardless of the result of the seismic test, the wildcatter receives the same cash flow stream (with actual NPV \$40,000), and yet the effective NPVs depend on the result of the seismic test. If the seismic test reveals a closed structure, the effective NPV is \$30,144, but if the test reveals an open structure the effective NPV is only \$16,304. This difference in effective NPVs reflects differences in information: though the seismic test result does not affect the cash flows directly, it does impact the wildcatter's consumption decisions. If the test reveals a closed structure, the wildcatter increases his consumption in period 1 in anticipation of receiving a large cash flow in

¹⁰ The effective NPVs, like the PCEVs, are independent of the timing of receipts of the cash flows in that, given the NPVs in each scenario, the timing of the cash flows itself conveys no further information about the utility derived from the project. As discussed earlier, the consumption streams will be identical regardless of the timing of receipt of the cash flows.

¹¹ This is why there are more points in the effective risk profile for the testing strategy in Figure 2 than there are in the actual risk profile for this strategy.

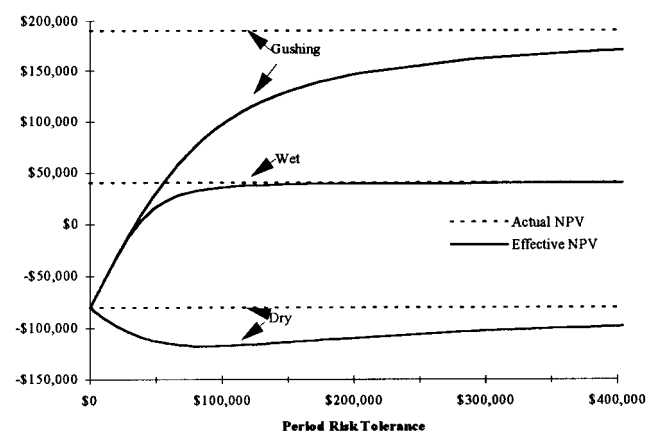
period 2. If the test reveals an open structure, he is less optimistic about the prospects of finding oil and does not increase his consumption in period 1, and *ex post* is worse off in terms of total utility for having not consumed more in period 1. Of course, the situation is reversed in the case of a dry hole: If the seismic test reveals a closed structure and the well turns out to be dry, the effective NPV is $-\$116,874$ versus an effective NPV of $-\$95,260$ when the test reveals an open structure. Thus the decision-maker's expectations in early periods affect his realized utilities.

We indicated earlier that an effective risk profile could be interpreted as a "net present gamble" equivalent to a project. One nice property of the effective risk profiles (as compared to actual risk profiles) is that they can be compared using stochastic dominance techniques. If, for example, the cumulative effective risk profile for one project lies to the right of the effective risk profile of another project (i.e., as in first-order stochastic dominance), we know that the first project is preferred to the second. These stochastic dominance results are, however, much weaker in this dynamic (or temporal) context than the corresponding results for static gambles that are resolved immediately. In the static case, if we compare two gambles and find that one dominates the other, we can conclude that all decision-makers in some class (i.e., all decision-makers preferring more money to less for first-order stochastic dominance) would prefer the dominating gamble. With dynamic gambles, the decision-maker's preferences play a role in determining the effective NPVs and, if you change preferences, you change effective risk profiles. In fact, it is possible to have two projects where one risk profile dominates another for one set of preferences and the dominance is reversed for a different set of preferences. The effective risk profiles do, however, provide an accurate depiction of the risks associated with a project—the values in the effective risk profile describe the overall utility actually realized in each scenario—and this is the reason risk profiles are usually calculated.

4.4. Sensitivity to Risk Attitude

Unlike present certainty equivalent values (PCEVs), effective NPVs may either increase or decrease with increasing risk aversion. We illustrate these results in Figure 4 by plotting the actual and effective NPVs for the

Figure 4 Actual and Effective NPV as a Function of Period Risk Tolerances



three possible outcomes given a "closed structure" test result as a function of the period risk tolerances. Here we assume (as in Figure 3) an additive-exponential utility with period risk tolerances that are all equal. In Figure 4, we see that as the period risk tolerances all increase, the effective NPVs all approach their actual counterparts, and that, as the risk tolerances all approach zero, the effective NPVs all approach the actual NPV of the worst possible outcome. In the limiting case of zero period risk tolerances, the decision-maker consumes in the first two periods assuming the well will be a dry hole, and then, if he actually strikes oil, he receives little additional utility (in the limit, no additional utility) from the increased consumption in the final period. The effective NPVs for the two good outcomes (wet or gushing) increase with increasing risk tolerance, while the effective NPV for the bad outcome decreases for a while and then begins to increase. Thus, in the exponential case and in general, the effective NPVs for individual outcomes may increase or decrease with increasing risk aversion.

5. Graphical Analysis

To better understand effective and actual NPVs and their relationships to the different notions of present certainty equivalents, we present a graphical illustration of a simple two-period, two-outcome gamble where these relationships can be made clear. We begin with a discussion of the relationship between actual and

effective NPVs in §5.1 and consider the relationship to present value certainty equivalents and PCEVs in §5.2.

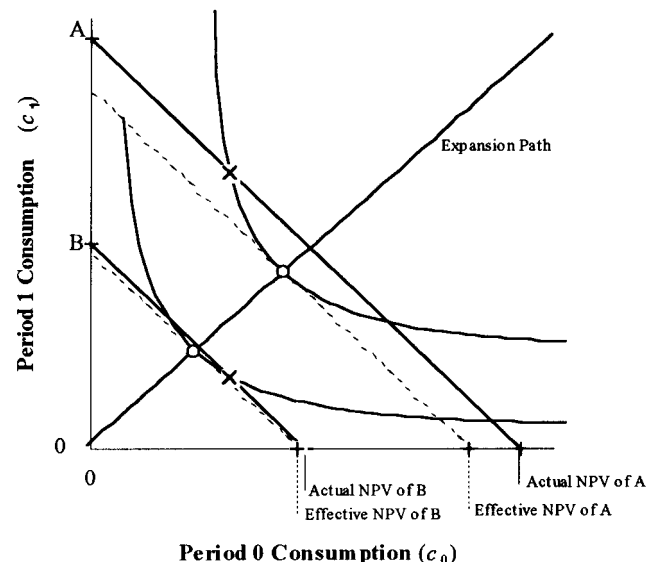
5.1. Actual and Effective NPVs

The relationship between actual and effective NPVs is illustrated in Figure 5. Here we consider a decision-maker who has zero initial wealth but owns the rights to a gamble resolved and paid in period 1 with equal probability of paying amount A or B . The feasible consumption paths are illustrated with downward sloping solid lines in Figure 5. In period 0, the decision-maker selects c_0 and thereby picks a vertical slice where his consumption will lie. In period 1, the uncertainty is resolved and the decision-maker winds up on either the top or the bottom line at the point corresponding to the selected c_0 . The slope of these lines is equal to $-(1 + r_f)^{-1}$ reflecting the interest paid on amounts borrowed in period 0; the consumption pairs on each line all have the same present value. The intersections of these lines with the vertical axis represents the possibility of consuming all of this income in period 1 (in which case the decision-maker consumes either A or B in period 1 and zero in period 0). The intersections of these lines with the horizontal axis represent the actual NPVs ($A/(1 + r_f)$ or $B/(1 + r_f)$) of the possible period 1 receipts.

Let us suppose that the optimal consumption strategy is the pair of points marked by x in the figure. Through these two points, we have drawn indifference curves representing consumption pairs with overall utility equal to the realized utility in the two outcomes of the gamble. Because we have assumed the decision-maker's utility function for consumption (U) is increasing in each argument and strictly convex, these indifference curves reflect increasing utility as we move upwards and towards the right and the indifference curves themselves are concave. The overall utility for a consumption pair ($U(c_0, c_1)$) represents a third dimension on this plot, rising out of the page.

Now suppose the decision-maker possesses some wealth in period 0, but has no additional income in period 1. The upward sloping line in Figure 5 labeled the "expansion path" (sometimes called "Engel curves" in microeconomics) represents the optimal consumption pairs as a function of the decision-maker's initial wealth, assuming no additional income in period 1. If, for ex-

Figure 5 Relationship Between Actual and Effective NPVs



ample, the decision-maker had a total wealth of \$100, he would consume at a pair of points (c_0^*, c_1^*) somewhere on this line. If he had a total wealth of \$200, he would consume at some higher point on this line. In general, these expansion paths need not be straight lines (though they are for additive exponential utilities); we have assumed this to be the case to simplify the drawing. The intersection of the indifference curves with this expansion path (marked with \circ in the figure) represents a scenario with utility equal to the utility derived from the particular outcome of the gamble but where the wealth is known in period 0 before making the consumption decision. The present value equivalent of this scenario is indicated by the intersection of the dashed downward sloping lines (with slope $(1 + r_f)^{-1}$) with the horizontal axis—these are the effective NPVs for the two scenarios. If the decision-maker received the effective NPV in period 0, he would consume at the points marked with the \circ and receive overall utility equal to the utility derived by the gamble with delayed resolution (with consumption at the points marked with x).

This graphical analysis provides some additional insights into the analysis of the previous section. Because the points on the expansion paths represent optimal consumption pairs given a certain amount of income in period 0, the dashed lines indicating consumption pairs

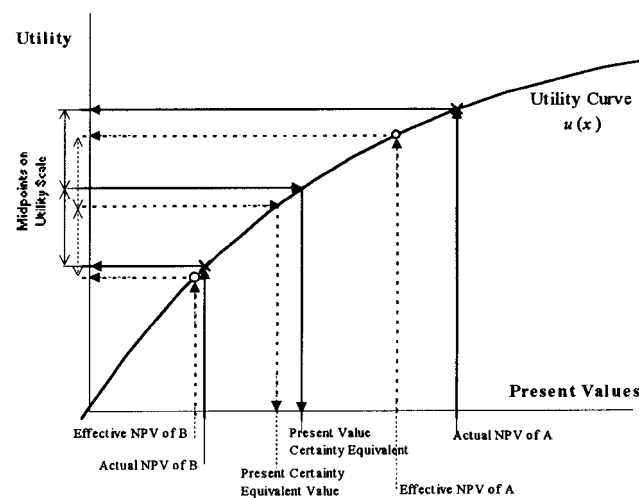
with the same present value as the points marked with \circ 's must be tangent to the indifference curves at these points. Given this, and the strict convexity of the indifference curves, we see that the effective NPVs will always be less than their actual counterparts. We also see that the shape of the indifference curves determines the magnitudes of the delay premium for a given scenario. If the indifference curves intersect the expansion paths at nearly right angles (i.e., are "L"-shaped), the effective NPVs will be significantly less than their actual counterparts. In this case, the decision-maker has very precise preferences for the allocation of consumption between the two periods and pays a heavy premium for not knowing the outcome of the gamble in period 1. In contrast, if the indifference curves are nearly parallel to the downward slope present value equivalent lines, the actual and effective NPVs will be close. In this case, gambles with delayed uncertainties are not appreciably different from gambles with immediate resolution and the ability to tailor consumption to the particular scenario is of little value.

5.2. Present Value Certainty Equivalents and Present Certainty Equivalent Values

To relate these actual and effective NPVs to project values, we consider the plot shown in Figure 6. Here we take a cross section of the plot in Figure 5 and show the overall utility $U(c_0, c_1)$ for points along the expansion path. The points on the expansion path are indexed by their present value equivalents; for example, the utility associated with the two effective NPVs in Figure 6 corresponds to the overall utility associated with the points marked with \circ in Figure 5. This utility curve represents the decision-maker's derived utility for period 0 wealth assuming no additional income, i.e., $u(v) = \mathcal{U}(v, 0, \dots, 0)$ where \mathcal{U} is the derived utility function specified in Equation (1). As argued in §3.4, this would be the utility curve an analyst would assess if assessing risk preferences using gambles resolved and paid in period 0.

The PVCEs and PCEVs are related to the actual and effective NPVs (respectively) through this derived utility function (u) in the same way that certainty equivalents are usually related to outcomes of a gamble. To determine the PVCE, we map the actual NPVs through the utility curve to their corresponding utilities, following the solid lines emanating from the horizontal axis

Figure 6 Calculation of Present Value Certainty Equivalents and Certainty Equivalent Present Values



in Figure 5. We then average these utility values according to their probabilities (here the average value is the midpoint between the two utilities since the two outcomes are equally likely) and map this value back through the utility function to find the PVCE. This is the certain amount that yields utility equal to the expected utility provided a 50-50 gamble involving the actual NPVs. The PCEV is given by the same process, following the dashed lines rather than the solid lines. Here we see that, because the effective NPVs are less than their actual counterparts, the PCEVs must be less than the corresponding PVCE. The delay premium for the gamble—given by the difference between the PVCE and the PCEV—thus depends on the difference between the effective NPVs and their actual counterparts and the curvature of the utility function u .

6. Summary and Conclusions

Most important decision problems—virtually all capital investments and planning situations—involve risky cash flows with uncertainties that are resolved over time. In most of these problems, the decision-maker has access to financial markets and may borrow and lend to smooth consumption over time. Yet, because of the difficulty of modeling these borrowing and lending decisions, these opportunities are rarely explicitly considered in decision and risk analyses and, unfortunately,

they cannot be implicitly modeled by applying expected utility techniques to income streams. In this paper, we have studied the errors induced by failing to model these borrowing and lending decisions and developed extensions to the standard decision and risk analysis procedures that take these borrowing and lending opportunities into account without overburdening the evaluation models. In these modifications, we discount cash flows at the risk-free rate for borrowing and lending but use different preferences to evaluate uncertainties resolved in different periods.

These new procedures are based on certain specific assumptions—additive exponential utilities, perfect markets for borrowing and lending, a known risk-free interest rate—that, like most assumptions in engineering and economic models, are best viewed as approximations of the actual decision situation. We see the assumptions and techniques of this paper playing a role in dynamic decision problems involving income streams much like the role that the exponential utility function has traditionally played when considering static gambles. Just as the exponential utility function is commonly used to approximate risk preferences in static setting (see Howard 1988), the assumptions and techniques developed in this paper allow one to capture the effects of market opportunities for borrowing and lending without explicitly modeling borrowing and lending decisions or assessing a full utility function for consumption.

While there are certainly situations where the additive-exponential assumption would be inappropriate, these methods may be used to treat a wide range of individual and corporate time and risk preferences, particularly those where the exponential utility function is used to approximate preferences for present value gambles.¹² Moreover, because most decision-makers have access to financial markets, we believe that these methods should be used instead of the standard procedures that neglect background trading opportunities. As dem-

¹² As a first approximation, it may be sufficient to assume an infinite horizon ($T = \infty$) and assume that the period risk tolerances (ρ_t) are constant over time. In this case, we can assess the current effective risk tolerance (R_0) using standard assessment techniques involving current gambles and then use effective risk tolerances of $R_t = R_0 / (1 + r_f)^t$ for later periods.

onstrated in the wildcatter example, the errors caused by failing to account for background borrowing and lending decisions can be significant, particularly when the project risks are substantial (compared to the decision-maker's risk tolerance) and there are long delays before the uncertainties are resolved. In these cases, the definitions and procedures developed in this paper may generate significantly different values and risk profiles and, consequently, different policies and insights.¹³

¹³ The author is grateful for helpful comments provided by Bob Clemen, Kevin McCardle, Bob Nau, Bob Winkler, and the anonymous associate editor and referees. This work was supported in part by the National Science Foundation under grant SBR-9511364.

Appendix

Proofs

We begin by proving Equation (8) as we use this result in some of the other proofs.

PROOF OF EQUATION (8). We first check the feasibility of the consumption stream specified by (8). To do this we note the amounts consumed at time t depend only on information available at time t and show that, in each scenario, the present value (discounting at the risk-free rate) of the amounts consumed is equal to the decision-maker's wealth (β_{-1}) plus the present value of the amount of income received. The present value of the amount consumed is equal to

$$\sum_{t=0}^T \frac{c_t^*}{(1+r_f)^t} = \sum_{t=0}^T \frac{c_t^0}{(1+r_f)^t} + \sum_{t=0}^T \left(\frac{\rho_t}{(1+r_f)^t} \sum_{\tau=0}^t \frac{w_\tau}{R_\tau} \right).$$

The first term on the right is equal to the decision-maker's wealth (β_{-1}) by the definition of c_t^0 . The second term on the right can be rewritten as

$$\sum_{t=0}^T \left(\frac{\rho_t}{(1+r_f)^t} \sum_{\tau=0}^t \frac{w_\tau}{R_\tau} \right) = \sum_{\tau=0}^T \frac{w_\tau}{R_\tau} \left(\sum_{t=\tau}^T \frac{\rho_t}{(1+r_f)^t} \right) = \sum_{\tau=0}^T w_\tau = v_T.$$

The first equality follows from exchanging the order of summation and the second from noting the inner summation is equal to the definition of the effective risk tolerance R_τ . The final equality follows from the definition of the windfalls and noting that the sum of the windfalls along a particular path through a tree is equal to the actual NPV for the scenario.

Next we show that the consumption stream specified by (8) is optimal. Given no project income and an additive exponential utility function, the decision-maker's optimization problem can be written as:

$$\max_{(\beta_0, \beta_1, \dots, \beta_{T-1})} - \sum_{t=0}^T k_t \exp \left(- \frac{(\beta_{t-1} - \beta_t)(1+r_f)^t}{\rho_t} \right).$$

The first-order necessary and sufficient conditions for this optimiza-

tion problem imply that, for all $t < T$ and all states of the world, the $c_t^0 = (\beta_{t-1} - \beta_t)(1 + r_j)^t$ satisfy:

$$k_t \frac{(1 + r_j)^t}{\rho_t} \exp(-c_t^0 / \rho_t) = k_{t+1} \frac{(1 + r_j)^{t+1}}{\rho_{t+1}} \exp(-c_{t+1}^0 / \rho_{t+1}). \quad (A1)$$

The corresponding first-order conditions given an uncertain income stream require that the c_t^* satisfy:

$$k_t \frac{(1 + r_j)^t}{\rho_t} \exp(-c_t^* / \rho_t) = k_{t+1} \frac{(1 + r_j)^{t+1}}{\rho_{t+1}} E_t[\exp(-\tilde{c}_{t+1}^* / \rho_{t+1})], \quad (A2)$$

where $E_t[-]$ denotes expectations taken at time t . We now show that with c_t^* given by (8), (A1) implies (A2). Using (8), (A2) becomes

$$\begin{aligned} & k_t \frac{(1 + r_j)^t}{\rho_t} \exp(-c_t^0 / \rho_t) \exp\left(-\sum_{\tau=0}^t w_\tau / R_\tau\right) \\ &= k_{t+1} \frac{(1 + r_j)^{t+1}}{\rho_{t+1}} \exp(-c_{t+1}^0 / \rho_{t+1}) \\ & \cdot \exp\left(-\sum_{\tau=0}^t w_\tau / R_\tau\right) E_t[\exp(-\tilde{w}_{t+1} / R_{t+1})], \end{aligned}$$

(A2) implies the leading terms on each side cancel and (A2) reduces to

$$0 = E_t[\exp(-\tilde{w}_{t+1} / R_{t+1})].$$

Using the definition of the period $t + 1$ windfall ($w_{t+1} = v_{t+1} - v_t$), this can be rewritten as

$$\exp(-v_t / R_{t+1}) = E_t[\exp(-\tilde{v}_{t+1} / R_{t+1})],$$

which follows from the definition of v_t as the effective certainty equivalent in Equation (4). \square

PROOF OF ROLLBACK PROCEDURE OF §3.3. To establish the validity of the rollback procedure for calculating PCEVs, we need to show that the expected utility derived from the project is equal to the utility derived from the lump-sum receipt of v_0 in period 0, as specified in Equation (2). From Equation (8), the utility derived from a lump-sum receipt of v_0 in period 0 is equal to

$$-\exp(-v_0 / R_0) \sum_{t=0}^T k_t \exp(-c_t^0 / \rho_t), \quad (A3)$$

and, using the fact that $w_0 = v_0$, the expected utility derived from the project is equal to

$$-\exp(-v_0 / R_0) \sum_{t=0}^T k_t \left(\exp(-c_t^0 / \rho_t) E_0 \left[\exp\left(-\sum_{\tau=1}^t \tilde{w}_\tau / R_\tau\right) \right] \right). \quad (A4)$$

To show that (A3) and (A4) are equal, note that, from the definition of the period t windfall ($w_t = v_t - v_{t-1}$),

$$E_t[\exp(-\tilde{w}_t / R_t)] = E_t[\exp(-\tilde{v}_t / R_{t+1})] \exp(v_{t-1} / R_{t+1}) = 1,$$

with the second inequality following from the definition of v_{t-1} as the effective certainty equivalent in Equation (4). Taking expectations iteratively, this implies

$$E_0 \left[\exp\left(-\sum_{\tau=1}^t \tilde{w}_\tau / R_\tau\right) \right] = E_0 \left[E_1 \left[\cdots E_t \left[-\sum_{\tau=1}^t \tilde{w}_\tau / R_\tau \right] \cdots \right] \right] = 1,$$

which implies (A4) is equal to (A3). \square

PROOF OF RISK SENSITIVITY RESULT OF §3.6. We first show that increasing risk aversion in the sense of §3.6 leads to lower PCEVs and then discuss the analogous results for PVCEs. Suppose that the utility function U^A is more risk averse than the utility function U^B in the sense discussed in §3.6. Let $(\tilde{c}_0^A, \tilde{c}_1^A, \dots, \tilde{c}_T^A)$ be the optimal consumption stream for a project using the utility function U^A . Let $(\tilde{c}_0^B, \tilde{c}_1^B, \dots, \tilde{c}_T^B)$ denote the certainty equivalent consumption streams defined such that each \tilde{c}_t^A satisfies $u_t^A(\tilde{c}_t^A) = E_0[u_t^A(\tilde{c}_t^A)]$. Using the assumed additivity of U^A , we have

$$E_0[U^A(\tilde{c}_0^A, \tilde{c}_1^A, \dots, \tilde{c}_T^A)] = \sum_{t=0}^T u_t^A(\tilde{c}_t^A).$$

Since the decision-maker can generate this certainty equivalent consumption stream (and thus achieve the same overall expected utility as derived from the project), given its present value

$$\sum_{t=0}^T \frac{\tilde{c}_t^A}{(1 + r_j)^t}, \quad (A5)$$

in wealth in period 0, the PCEV of the project under utility U^A must be equal to this amount less the decision-maker's initial wealth.

Let $(\tilde{c}_0^B, \tilde{c}_1^B, \dots, \tilde{c}_T^B)$ be the certainty equivalent stream for utility U^B corresponding to the consumption stream $(\tilde{c}_0^A, \tilde{c}_1^A, \dots, \tilde{c}_T^A)$ that is optimal for utility U^A , so that $u_t^B(\tilde{c}_t^B) = E_0[u_t^B(\tilde{c}_t^A)]$. Because each period utility function u_t^A is more risk-averse than the corresponding period utility function u_t^B , we have $\tilde{c}_t^B \geq \tilde{c}_t^A$ (see Pratt 1964, Theorem 1) and thus the present value of $(\tilde{c}_0^B, \tilde{c}_1^B, \dots, \tilde{c}_T^B)$,

$$\sum_{t=0}^T \frac{\tilde{c}_t^B}{(1 + r_j)^t}, \quad (A6)$$

exceeds that of $(\tilde{c}_0^A, \tilde{c}_1^A, \dots, \tilde{c}_T^A)$ given by (A9). Because the consumption stream $(\tilde{c}_0^A, \tilde{c}_1^A, \dots, \tilde{c}_T^A)$ is feasible for utility U^B , but not necessarily optimal, the PCEV under utility U^B exceeds (A6), which exceeds (A5), which is equal to the PCEV under U^A . Thus, increasing risk aversion in the sense of §3.6 leads to a decrease in PCEVs. The result for PVCEs follows from the result for PCEVs since the PVCE of a project can be viewed as the PCEV for a project with the same probabilities and pay-offs, but with all uncertainties resolved in period 0. \square

PROOF OF EQUATIONS (9), (10) AND (11). The utility derived from no income (U^0) is equal to

$$-\sum_{t=0}^T k_t \exp(-c_t^0 / \rho_t). \quad (A7)$$

From the first-order conditions for the decision-maker's optimization problem given no income (see Equation (A1) above), there exists a constant λ such that, for all t ,

$$-k_t \exp(-c_t^0 / \rho_t) = \lambda \frac{\rho_t}{(1 + r_j)^t}. \quad (A8)$$

Substituting this into (A7), we have

$$U^0 = \lambda \sum_{t=0}^T \rho_t / (1 + r_f)^t = \lambda R_0. \quad (A9)$$

Using (8) and (A8), the utility derived from a windfall stream (w_0, w_1, \dots, w_t) is equal to

$$\begin{aligned} & - \sum_{t=0}^T k_t \left(\exp(-c_t^0 / \rho_t) \exp\left(-\sum_{\tau=0}^t w_\tau / R_\tau\right) \right) \\ & = \lambda \sum_{t=0}^T \left(\frac{\rho_t}{(1 + r_f)^t} \exp\left(-\sum_{\tau=0}^t w_\tau / R_\tau\right) \right), \end{aligned}$$

which, using (A9), implies Equation (9). Equation (10) then follows as described in the text. From Equation (10) to Equation (11) is algebra. Pulling out the $t = 0$ term from the summation of Equation (10), we can rewrite (9) as $\lambda = U^0 / R_0$ times

$$\frac{\rho_0}{(1 + r_f)^0} \exp(-w_0 / R_0) + \sum_{t=1}^T \left(\frac{\rho_t}{(1 + r_f)^t} \exp\left(-\sum_{\tau=0}^t w_\tau / R_\tau\right) \right).$$

Adding $\sum_{t=1}^T (\rho_t / (1 + r_f)^t) \exp(-w_0 / R_0)$ to first term here and subtracting it from the second, this becomes

$$\begin{aligned} & R_0 \exp(-w_0 / R_0) \\ & + \sum_{t=1}^T \frac{\rho_t}{(1 + r_f)^t} \left(\exp\left(\sum_{\tau=1}^t -w_\tau / R_\tau\right) - 1 \right) \exp(-w_0 / R_0). \end{aligned}$$

Rewriting the first term to put it in the form of Equation (4), this becomes

$$\begin{aligned} & R_0 + R_0(\exp(-w_0 / R_0) - 1) \\ & + \sum_{t=1}^T \frac{\rho_t}{(1 + r_f)^t} \left(\exp\left(\sum_{\tau=1}^t -w_\tau / R_\tau\right) - 1 \right) \exp(-w_0 / R_0). \end{aligned}$$

Now, adding $\sum_{t=2}^T (\rho_t / (1 + r_f)^t) \exp(-w_1 / R_1)$ to first term in the summation and subtracting it from the remaining terms, we can rewrite this expression as:

$$\begin{aligned} & R_0 + \sum_{t=0}^1 \left(R_t(\exp(-w_t / R_t) - 1) \exp\left(\sum_{\tau=0}^{t-1} -w_\tau / R_\tau\right) \right) \\ & + \sum_{t=2}^T \frac{\rho_t}{(1 + r_f)^t} \left(\exp\left(\sum_{\tau=1}^t -w_\tau / R_\tau\right) - 1 \right) \exp\left(\sum_{\tau=0}^1 -w_\tau / R_\tau\right). \end{aligned}$$

Continuing in this manner, pulling terms from the second summation to the first, we find Equation (10) is equal to $\lambda = U^0 / R_0$ times

$$R_0 + \sum_{t=0}^T \left(R_t(\exp(-w_t / R_t) - 1) \exp\left(\sum_{\tau=0}^{t-1} -w_\tau / R_\tau\right) \right),$$

which implies equation (11) of the text. \square

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