Fisher Separation and Project Valuation in Partially Complete Markets

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Abstract

This paper presents an extension of the Fisher Separation Theorem applicable in multiperiod, "partially complete" markets where some, but not all, risks may be hedged by trading securities. Given necessary and sufficient conditions on preferences, an investor contemplating investments in productive opportunities (or projects) and financial securities can decompose this problem into production and portfolio-consumption problems that may be solved sequentially. In the production problem, the investor evaluates alternative production plans using a dynamic programming-like procedure that integrates contingent-claims techniques and the recursive utility approach developed by Kreps and Porteus, using market prices to value market risks and the investor's beliefs and preferences to value private risks. In the portfolio-consumption problem, the investor can ignore the project details and choose a trading strategy to maximize his expected utility of consumption, given an initial wealth reflecting the maximal project value. The valuation procedure can be used to generate bounds on project values when the preference assumptions are not exactly satisfied and can be used with aggregate investor beliefs and preferences in the case of multiple investors sharing a project.

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The Fisher Separation Theorem (Fisher 1930) stands as one of the cornerstones of modern corporate finance, providing a justification for both the "NPV rule" and the separation of ownership and management. Originally stated for a deterministic world with perfect markets, the separation theorem says that an investor considering both productive opportunities (called *projects*) and market opportunities for borrowing and lending can decompose this *grand* consumption-portfolio-production problem into simpler *production* and *portfolio-consumption* problems that may be solved sequentially. In the production problem, the investor chooses among alternative production plans to maximize the present value of the income generated. In the portfolio-consumption problem, the investor chooses a borrowing and lending strategy that maximizes his utility for consumption given that his initial wealth has been increased by the maximal project value.

A key feature of this separation result is that the solution to the production problem depends only on objective information (the project's cash flows and the market rate for borrowing and lending) and can be determined independently of the investor's subjective preferences for consumption. Thus investors with diverse preferences may cooperate in projects and delegate production decisions to a manager who need not know anything about the preferences of the investors. The manager's task is to maximize the present value of the project and all investors will agree on the appropriate course of action.

This separation theorem generalizes perfectly to the case of uncertainty provided markets are complete in that every project risk can be perfectly hedged by trading existing securities. This was noted in the classic works of Arrow (1964) and Debreu (1959) and developed more fully by Hirshleifer (1965) and Drèze and Modigliani (1972). In this setting, the *present certainty-equivalent value* plays the role of present value in the deterministic case and is given by the current market value of a portfolio of securities that exactly matches the project's value at all times and in all states. The solution of the production problem is again independent of the investor's beliefs and preferences and production decisions may still be delegated to managers with no loss.

The assumption of complete markets required for this separation theorem is quite strong and limiting in many contexts. While there exist well-organized and efficient markets for some risks, "once it is realized how unimaginably numerous is the set of all distinguishable states of the world, and that markets cannot be provided without cost, we are forced to the conclusion that in the real world markets

will necessarily be incomplete" (Hirshleifer 1970, pg. 273). Having recognized the failure of separation in incomplete markets in general, many researchers have identified special cases where separation holds in incomplete market settings. For example, Diamond (1967) develops a two-period model with a risk-neutral firm whose production function can be decomposed into the product of its input and some uncertain risk that can be hedged by trading securities; he finds that the firm's production decisions are entirely determined by market prices. Similarly, Baron (1976) develops a two-period model of firms facing exchange rate risks. Here, though markets are incomplete, each producer's risks can be perfectly hedged using the available futures contracts and, consequently, firms wind up bearing no residual risks and production decisions depend only on market prices. Like these two examples, most of the literature in this vein has focused on identifying assumptions about the production technology that lead to separation and/or the case where the securities market spans the production uncertainties. DeAngelo (1981) and Drèze (1982) provide syntheses of the early literature in this area and Kamara (1993) provides some more recent references.

The goal of this paper is to present a generalization of the Fisher Separation Theorem applicable in the case where the investor winds up holding some residual risks. Though we make no specific assumptions about project or security returns, we find that we must place some restrictions on the investor's preferences and on the form of incompleteness in the market. First, we must assume that the investor's preferences for consumption can be captured by a time-additive, exponential utility function. Second, we must assume that the market is "partially complete" in that there exists an efficient, complete securities market embedded in a richer model that includes unhedgable "private" as well as hedgable "market" risks; the private risks need not be independent of market risks. Given these assumptions, the investor considering both projects and market opportunities for trading securities can decompose this grand problem into simpler production and portfolio-consumption problems that may be solved sequentially in exactly the same way as in the deterministic and complete markets cases.

With incomplete markets, it is impossible to determine unique project values independent of the investor's beliefs and preferences with incomplete markets. However, in this framework the investor's choice of production plans depends on his beliefs about private risks and his risk preferences, but, as in complete markets, the solution is independent of his beliefs about market risks and his time preferences. The solution to the portfolio-consumption problem depends on the investor's beliefs about market risks

and his time preferences, but does not depend on the details of the project beyond its present certainty equivalent value. The investor's grand strategy for investing in securities is given by the sum of a speculative strategy found in the portfolio-consumption problem and *ex ante* and *ex post* risk management strategies identified as a byproducts of solving the production problem. The *ex ante* risk management strategy is for optimally reducing project risks before the uncertainties are resolved and the *ex post* strategy is for optimally managing or "rebalancing" the trading strategy after resolution of the unhedgable private uncertainties.

The valuation procedure associated with this separation theorem is a natural extension of the contingent-claims procedures used in complete markets. In one approach, we determine project values using a generalization of the replicating arguments used in complete markets. In this approach, we use the investor's subjective beliefs and preferences to calculate market-state-contingent project values – essentially projecting them onto a complete markets subspace – and then construct a portfolio and trading strategy that matches these values in every market state. Alternatively and equivalently, we can value projects using an extension of "risk-neutral" pricing methods (Cox and Ross 1978, Harrison and Kreps 1979). In this approach, we value market risks by taking expectations using risk-neutral probabilities and value private risks using the investor's probabilities and utilities. From this perspective, the valuation procedure may also be seen as an extension of the recursive utility (or temporal von-Neumann-Morgenstern) procedure developed in Kreps and Porteus (1978, 1979a). In the case where the only security available is the risk-free security, the valuation procedure reduces to a special form of the Kreps-Porteus procedure. More generally, the valuation procedure can be viewed as an extension of the Kreps-Porteus procedure that uses market information (in the form of risk-neutral probabilities) to value market risks.

The idea of valuing projects or cash flows in incomplete markets by projecting them onto the marketed subspace is fairly standard, playing a central role in Merton (1998) and discussed in detail in Magill and Quinzi (1996). The method described here requires the consideration of the investor's risk-preferences to determine how the project cash flows are projected onto the market subspace. Merton's projection minimizes the squared error in the hedge portfolio and he assumes risk-neutrality towards the unhedged, residual risk. Magill and Quinzi's projections are sensitive to investors' risk preferences but produce values corresponding to an infinitesimal or marginal investment in the income stream generated

by the project (Magill and Quinzi 1996; pg 152). Luenberger (1996) calls these values "zero-level prices" since they describe the price such the investor would neither want to buy or sell a share of the project. Here the values correspond to the purchase or sale of the entire project (i.e., present certainty equivalent values) and, unlike these other situations, the projection used here is not linear or additive. These different notions of value coincide in the limiting case of a risk-neutral investor.

The motivation for this paper comes from the study of alternative methods for solving real options problems. Traditionally, in solving these problems, one either assumes that projects are spanned by the securities market and uses contingent-claims methods or, alternatively, uses dynamic programming methods without explicitly modeling any related trading decisions and using an exogenously specified discount rate (see Dixit and Pindyck 1994 and Smith and Nau 1995). Yet, in most real options problems encountered in practice, some, but not all, project risks can be hedged by trading existing securities. For example when evaluating an oil property, price risks can be hedged (at least in part) using oil futures contracts, but project-specific risks like production rates or drilling costs cannot be hedged by trading existing securities. In problems like these, it is inappropriate to assume spanning but difficult to explicitly model all of the relevant beliefs, preferences, and trading decisions in the dynamic programming framework. The valuation procedure developed here is like the contingent-claims procedure in that it implicitly takes into account market opportunities to trade without complicating the model, but, unlike the standard contingent-claims procedure, it does not require complete markets. Smith and McCardle (1998) use this valuation procedure to evaluate an oil field where the investor may accelerate production or abandon the property at any time; oil prices and production rates are both stochastic and prices can be hedged by trading futures contracts.

The analysis of this paper is carried out in a discrete-time, discrete-space model. We begin by considering the perspective of a single investor whose goal is to maximize his expected utility of consumption by invest in traded securities and non-traded projects. The investor takes the set of securities and their prices as given. The analysis is thus a partial equilibrium analysis and one might interpret the results or the paper as applicable to an investor who is "small" in that his actions do not affect market prices or market structure.[1] This basic framework is described in Section 1 along with a

[1] For a discussion of models of equilibrium in incomplete markets, the reader is referred to the surveys of Duffie (1992), Magill and Shafer (1991), and Geanakoplos (1990) and the text Magill and Quinzi (1996). For a recent review of the "financial market innovation" literature, see Allen and Gale (1994) or Duffie and Rahi (1995).

simple numerical example that will be used to illustrate the procedures and results developed later in the paper.

In Section 2, we state the market and preference restrictions required for the separation theorem and, in Section 3, we describe the valuation procedure used to solve the production problem. In Section 4, we study the investor's securities trading decisions and show how the investor can decompose his grand problem into production and portfolio-consumption problems that may be solved sequentially; these results are summarized in a generalized version of the Fisher Separation Theorem. In Section 5, we consider extensions of the separation result and describe which aspects of the result fail as we relax each of the necessary preference and market assumptions. We also present a simple approximation result that gives bounds on project values as we relax the exponential utility assumption.

In Section 6, we consider the case of multiple investors (a partnership) sharing projects and describe how one can separate ownership and control given partially complete markets. The main result of this section is a dynamic aggregation result where investors' heterogeneous beliefs about private risks are aggregated as in Wilson (1968) and used in the valuation procedure developed earlier. Again the project values are independent of the investors' beliefs about market risks and their time preferences. All proofs are given in an appendix.

1. The Basic Framework

Our model is a standard discrete-time, discrete-space model of a multiperiod economy where (until Section 6) we focus on modeling the decisions of a single agent, hereafter referred to as the *investor*.

1.1. Information Structure

The possible states of the world are described by a measure space $(\Omega, \mathcal{F})$ where Ω denotes the finite set of possible states of the world (ω) and $\mathcal{F}$ is the collection of all subsets of Ω with elements $A \in \mathcal{F}$ representing *events*. The investor's beliefs about the likelihood of the various possible events are described by a probability measure P defined on $(\Omega, \mathcal{F})$ that is assumed to be strictly positive in that $P(A) > 0$ for all non-null events $A \in \mathcal{F}$. All expectations (E[-]) are determined using P unless otherwise noted.

Uncertainties are resolved and trading takes place at times $t = 0,1,\ldots, T$. The information known to the investor is described by a sequence of algebras, $\mathcal{F}_t \subseteq \mathcal{F}$, for $t = 0,1,\ldots, T$. The interpretation is that the investor knows at time t whether or not $\omega \in A_t$, for each $A_t \in \mathcal{F}_t$. As usual, we will assume that the

algebras grow increasingly fine in that $\mathcal{F}_s \subseteq \mathcal{F}_t$, whenever $s \leq t$, meaning that events are never "forgotten." For convenience and without loss of generality, we take $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$ so there is no information revealed before period 0 or after period T. The sequence of algebras $\boldsymbol{F} = \{\mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_T\}$ will be referred to as the *investor's filtration*.

At several points in the paper, it will be convenient to identify states (or nodes) representing the investor's possible states of information at a particular time. To formalize this notion, for each $\mathcal{F}_t$, we may uniquely define a partition $\mathcal{P}_t$ generated by $\mathcal{F}_t$ as the smallest collection of disjoint non-empty subsets S_t (referred to as time-t states) of Ω whose union is Ω and is such that each $A_t \in \mathcal{F}_t$ may be represented as a union of states S_t. The interpretation is that the investor knows which state S_t prevails at time t. The requirement that the algebras grow increasingly fine translates here to the requirement that the time-(t+1) partition $\mathcal{P}_{t+1}$ include a set of states that partition each time-t state S_t, so the investor's state of information grows more refined over time. This equivalent representation of the information structure corresponds to an event tree with terminal states S_T being endpoints of the tree and earlier states S_t being nodes in the tree.

We will be concerned with modeling income streams, securities price processes, and trading strategies that will be represented as $\boldsymbol{F}$-adapted stochastic processes. This means that given a process $\boldsymbol{x} = (x_0, x_1, ..., x_T)$, we require each x_t to be a random variable with respect to $(\Omega, \mathcal{F}_t)$, so the investor knows (or can determine) the value of x_t at time t. We let $\boldsymbol{X}$ denote the space of all real-valued $\boldsymbol{F}$-adapted stochastic processes and note that $\boldsymbol{X} \subseteq \Re^{T+1} \times \Re^{|\Omega|}$ where $|\Omega|$ indicates the number of states ω in Ω.

1.2. Securities and Securities Markets

There are N+1 long-lived securities. For notational convenience, we will assume that the securities pay no dividends in the time frame of the model. We let $\boldsymbol{s}_t = (s_t^0, s_t^1, \ldots, s_t^N) \in \Re^N$ denote the vector of security prices in period t. The security price process $\boldsymbol{s} = (\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_T)$ is assumed to be exogenously determined and adapted to $\boldsymbol{F}$. We will assume that there is a risk-free security (the 0th security) with price $s_t^0(\omega) = (1+r_f)^t$ for all t and $\omega \in \Omega$, with r_f being the *risk-free rate*.

The market is linear and frictionless in that the investor can buy or sell as many shares of a security as desired (including fractional and negative amounts) at market prices without incurring any transactions costs. Let $\boldsymbol{\theta}_t = (\theta_{t0}, \theta_{t1}, \ldots, \theta_{tN})$ denote a generic portfolio of shares of securities held from time t to time t+1. As the investor's trading strategies must depend only on information known at time t, we require all

trading strategies $\boldsymbol{\theta} = (\boldsymbol{\theta}_0, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_T)$ to be adapted to $\boldsymbol{F}$, and, to prevent "borrowing from beyond the horizon", we require $\boldsymbol{\theta}_T = \mathbf{0}$; the space of all such trading strategies is denoted $\boldsymbol{\Theta}$. If the investor follows trading strategy $\boldsymbol{\theta}$, he receives proceeds $d(\boldsymbol{\theta}) \in \boldsymbol{X}$ given by $d_t(\boldsymbol{\theta}) \equiv (\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t)\, \boldsymbol{s}_t$ (the product here is an inner product), for $t > 0$, and $d_t(\boldsymbol{\theta}) \equiv -\boldsymbol{\theta}_t\, \boldsymbol{s}_t$ for $t = 0$.

The securities market is assumed to be *arbitrage-free* in that security prices are such that there is no trading strategy $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ that always generates non-negative proceeds ($d_t(\omega; \boldsymbol{\theta}) \geq 0$, for all t and ω) and has some chance of generating positive proceeds ($d_t(\omega; \boldsymbol{\theta}) > 0$, for some t and ω).

1.3. Projects

Projects are modeled as income streams that, unlike securities, are not traded. We imagine these projects or productive opportunities as endowments resulting from unique, investor-owned (or potentially owned) patents, land or natural resources, technical knowledge, etc. The impossibility of trading or sharing these projects might be the result of, for example, insurmountable transaction costs or moral hazard problems. We let $\boldsymbol{p} = (p_0, p_1, \ldots, p_T) \in \boldsymbol{X}$ denote a generic project and, when the investor has flexibility in selecting or managing a project, we let $\boldsymbol{p}(\boldsymbol{\pi}) \in \boldsymbol{X}$ denote the project cash flows given that the investor follows *production plan* $\boldsymbol{\pi}$.

To formalize the structure of these production plans and capture the possibility of sequential decision making, we let $\mathcal{A}_t$ denote the set of possible actions (α_t) at time t. A production plan $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_T)$ is sequence of policies π_t mapping from states of the world ($\omega \in \Omega$) to time-t actions ($\alpha_t \in \mathcal{A}_t$). To ensure that the choices are made based only on available information, we require the policies $\boldsymbol{\pi}$ to be adapted to the investor's filtration $\boldsymbol{F}$. Similarly, we require the project cash flows to be "non-anticipating" in that the period-t project cash flows do not depend on decisions that have not yet been made. Formally, this means that for any t, $p_t(\boldsymbol{\pi}^A) = p_t(\boldsymbol{\pi}^B)$ for any $\boldsymbol{\pi}^A$ and $\boldsymbol{\pi}^B$ such that $\pi^A_\tau = \pi^B_\tau$ for $\tau = 0, 1, \ldots, t$. To ensure that an optimal production plan exists, we will assume that $\boldsymbol{p}(\boldsymbol{\pi})$ is a continuous function of $\boldsymbol{\pi}$ and $\boldsymbol{\pi}$ ranges over some compact set of available plans $\boldsymbol{\Pi}$. (We can use the Euclidean topology for $\boldsymbol{X}$; the topology for $\boldsymbol{\Pi}$ will depend on the application.)

1.4. The Investor's Grand Problem

The investor's goal is to maximize his expected utility of consumption where consumption is modeled as the net income to the investor, taking into account both securities and projects. The

investor's preferences for a (realized) consumption stream $\boldsymbol{c} = (c_0, c_1, ..., c_T) \in \Re^{T+1}$ are captured by a utility function $U(c_0, c_1, ..., c_T)$, or simply $U(\boldsymbol{c})$, that is assumed to be strictly increasing and strictly concave in $\boldsymbol{c}$. The investor's *grand problem* (or consumption-portfolio-production problem) is to choose a production plan $\boldsymbol{\pi}^*$ and a securities trading strategy $\boldsymbol{\theta}_g$ to solve

$$\max_{\boldsymbol{\pi} \in \boldsymbol{\Pi},\, \boldsymbol{\theta}_g \in \boldsymbol{\Theta}} \quad \mathrm{E}[\, U(\boldsymbol{e} + \boldsymbol{p}(\boldsymbol{\pi}) + \boldsymbol{d}(\boldsymbol{\theta}_g))\,] \qquad , \tag{1}$$

where $\boldsymbol{e} = (e_0, 0, ..., 0)$ denotes the a cash flow stream corresponding to a lump-sum receipt of the investor's (endowed) initial wealth (e_0) at time 0. $\boldsymbol{e} + \boldsymbol{p}(\boldsymbol{\pi}) + \boldsymbol{d}(\boldsymbol{\theta}_g)$ is the consumption stream generated by the project and securities investments using production plan $\boldsymbol{\pi}$ and trading strategy $\boldsymbol{\theta}_g$. Note that in this formulation of the grand problem, we allow the possibility of negative consumption in any period. For reasons discussed in Section 5.3, we cannot fully separate production and portfolio-consumption decisions if we require consumption to be non-negative.

1.5. A Simple Example

To illustrate the results and procedures of the paper, we will consider a simple numerical example involving a farmer who is using a new seed and faces price and yield uncertainty. The farmer plants his crop in period 0 and observes some early indication the effectiveness of the new seed in period 1 which gives him some information about future yields. There is still uncertainty about the yield due to, say, uncertainty about the weather near harvest. In period 2, he harvests the crop and sells it at then-prevailing market prices. The cash flows and probabilities are shown in the tree of Figure 1. The farmer may buy and sell two securities: a risk-free security whose price in period t is given by $(1+.04)^t$ (corresponding to a risk-free rate r_f of 4 percent) and a forward contract that guarantees delivery of a certain amount of the crop for \$13.00. The contract requires no up front cash payments in periods 1 or 2 and pays the difference between the spot price and \$13, as shown in Figure 1. The farmer's preferences are captured by an additive-exponential utility function of the form described in Section 2.1 (below) with utility weights $k_0 = k_1 = k_2 = 1$ and consumption risk tolerances $\rho_0 = \rho_1 = \rho_2 = 40$. His initial wealth ($e_0$) is assumed to be 100.

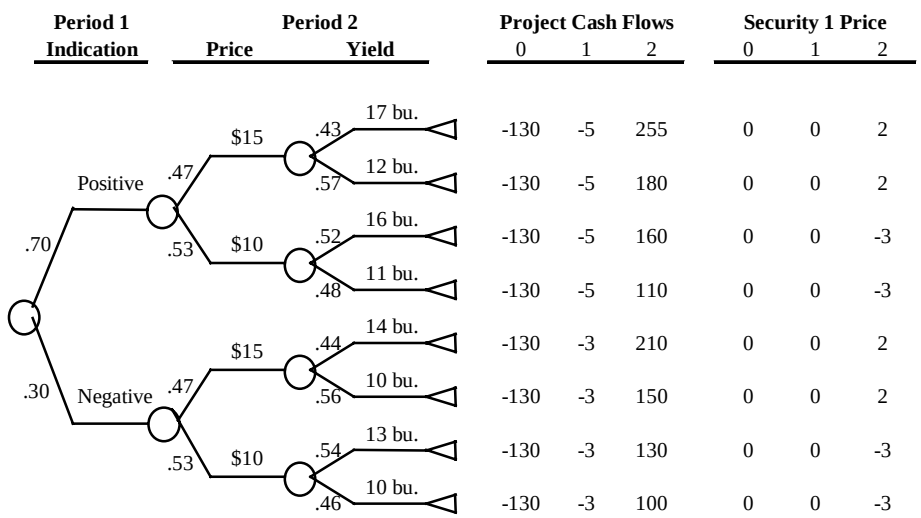

Figure 1: The farmer's problem.

2. Valuation in Partially Complete Markets

Given certain preference and market restrictions, we can decompose the grand problem (1) into simpler production and portfolio-consumption problems that focus exclusively on projects and exclusively on securities, respectively. After describing the preference and market restrictions in Sections 2.1 and 2.2, we describe the procedure for valuing a single project in the next section.

2.1. Preference Assumptions

To achieve separation with incomplete markets, we assume that the investor's preferences for consumption satisfy the following two assumptions.

A1) **Additivity:** *The investor's utility function can be represented as the sum of utility functions for consumption in individual periods,* $U(c_0, c_1, \ldots, c_T) = \sum_{t=0}^{T} k_t\, u_t(c_t)$ *where* $k_t > 0$ *for all t.*

A2) **Constant Absolute Risk Aversion:** *The investor's preferences for period-t consumption exhibit constant absolute risk aversion and, hence, can be represented by a utility function of the form* $u_t(c_t) = -\exp(-c_t/\rho_t)$ *where* $\rho_t > 0$, *for all t.*

The *utility weights* k_t and *consumption risk tolerances* ρ_t can be interpreted as capturing the investor's time and risk preferences, respectively. The second assumption implies that the investor is indifferent between an uncertain period-*t* consumption level $\tilde{c}_t$ and certain level given by the certainty equivalent,

$$\mathrm{CE}_t[\tilde{c}_t] \equiv -\rho_t \ln(\mathrm{E}[\exp(-\tilde{c}_t/\rho_t)]) \ .$$

These two assumptions (together with our assumptions about the continuity of $\boldsymbol{p}(\boldsymbol{\pi})$, the compactness of $\boldsymbol{\Pi}$, and the absence of arbitrage opportunities) ensure that there exists a solution to the grand problem.

Lemma 1: *If the investor's preferences satisfy (A1) and (A2), then there exists* $\boldsymbol{\pi}^*$ *and* $\boldsymbol{\theta}_g^*$ *that solve (1).*

To show how we will use these preference assumptions, let us consider a simplification of the grand problem (1) where the investor has no project income and the only security available is the risk-free security. In this case, given (A1) and (A2), the investor's grand problem (1) reduces to

$$U^* \equiv \max_x \sum_{t=0}^{T} -k_t \exp(-x_t/\rho_t)$$

subject to
$$\sum_{t=0}^{T} x_t/(1+r_f)^t = e_0 \ .$$

The constraint being that the present value of the consumption stream must equal to investor's initial wealth e_0. Letting μ_0^* denote the Lagrange multiplier associated with the constraint, the first order conditions for this problem require that the optimal solution $\boldsymbol{x}^*$ satisfy $-k_t\exp(-x_t^*/\rho_t) = \mu_0^* \rho_t/(1+r_f)^t$ for all t. Thus we may write the optimal utility as $U^* = \mu_0^* R_0$ where $R_0 \equiv \sum_{t=0}^{T} \rho_t/(1+r_f)^t$.

Now suppose the investor receives the additional project income amount p_0 in period 0. Taking μ_0^+ to be the new Lagrange multiplier, the first-order conditions require that the optimal solution $\boldsymbol{x}^+$ satisfy $-k_t\exp(-x_t^+/\rho_t) = \mu_0^+ \rho_t/(1+r_f)^t$ for all t. Taking $x_t^+ = x_t^* + p_0 \rho_t/R_0$, we see that $-k_t\exp(-x_t^+/\rho_t) = -k_t\exp(-x_t^*/\rho_t) = \mu_0^* \rho_t/(1+r_f)^t \exp(-p_0/R_0)$ for all t, so the first-order conditions are satisfied with $\mu_0^+ = \mu_0^* \exp(-p_0/R_0)$. Thus, given that $\boldsymbol{x}^*$ is the optimal consumption stream with no project income, $x_t^+ = x_t^* + p_0 \rho_t/R_0$ is optimal given this additional income in period 0. Moreover, the investor's utility is now $U^+ = U^*\exp(-p_0/R_0)$, implying that the investor's preferences towards period-0 income can be described by an exponential utility function with an "effective risk tolerance" equal to R_0. Thus, given a gamble $\tilde{p}_0$ that is resolved and paid in period 0, the investor is just indifferent between receiving the gamble and the "effective certainty equivalent" equal to $-R_0 \ln(\mathrm{E}[\exp(-\tilde{p}_0/R_0)])$. The investor spreads the outcome of the gamble over future periods by increasing consumption in each period by $\tilde{p}_0 \rho_t/R_0$.

This kind of intertemporal risk sharing and aggregation will play a key role in the results that follow. In the valuation procedure of Section 3, we will evaluate private risks (those that cannot be hedged by trading existing securities) by calculating these effective certainty equivalents. In the

separation theorem, we see that the optimal trading strategy contains a rebalancing component that spreads private risks over time by adjusting the position in the risk-free security. Intuitively, these risk sharing and aggregation results are analogous to a risk-sharing syndicate of Wilson (1968). Here, it is as if the investor forms a syndicate with his "future selves" and shares risks by investing in the risk-free security. Wilson shows that individuals with exponential utilities, as a Pareto efficient group, should behave as if they have an exponential utility with risk tolerance equal to the sum of the individual risk tolerances and should share risks in proportion to their individual risk tolerances. Here each "future self" has an exponential utility function for consumption in that period and the analogous results hold with discounted consumption risk tolerances, reflecting the interest earned on the risk-free security.

2.2. Market Assumptions

In addition to restricting the investor's preferences, to achieve separation we must assume that there is a complete, efficient securities market embedded within the multiperiod model. To formalize this idea, we introduce a *market filtration*, $\boldsymbol{F}^m \equiv \{\mathcal{F}_t^m : t = 0, 1, ..., T\}$, a sequence of increasingly fine algebras of Ω, with elements A_t^m of $\mathcal{F}_t^m$ referred to as *time-t market events*. We let the *market states* S_t^m and *market partitions* $\mathcal{P}_t^m$ be defined in the same way that the investor's states of information were defined in Section 1.1. At each time we require $\mathcal{F}_t^m \subseteq \mathcal{F}_t$; the interpretation is that the investor knows the market state but may also have private information. This implies that for each private state $S_t \in \mathcal{P}_t$ there is a unique market state $S_t^m \in \mathcal{P}_t^m$ such that S_t is a subset of S_t^m. In practice, we envision these market states as being defined by reference to the security price process, perhaps corresponding to a binomial- or trinomial-tree model (as in Cox, Ross, and Rubinstein 1979, for example). In the farmer's problem of Figure 1, the market filtration describes price changes in the prices of the forward contract and is illustrated in the tree Figure 2.

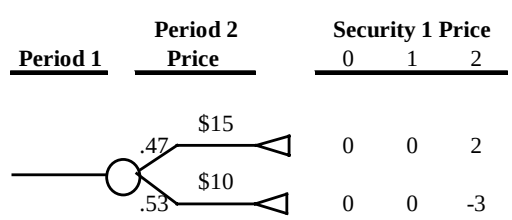

Figure 2: Market filtration for the farmer's problem

Let $\boldsymbol{\Theta}^m \subseteq \boldsymbol{\Theta}$ denote the set of trading strategies that are adapted to $\boldsymbol{F}^m$ and thus depend on market information only. The market assumptions may then be formalized as follows:

(A3) **Partial Completeness**: *The securities market is complete within* $\boldsymbol{F}^m$ *in that, for every* $\boldsymbol{x} \in \boldsymbol{X}$ *that is adapted to* $\boldsymbol{F}^m$*, there exists a replicating trading strategy* $\boldsymbol{\theta}_r \in \boldsymbol{\Theta}^m$ *such that* $d_t(\omega, \boldsymbol{\theta}_r) = x_t(\omega)$ *for all* $\omega \in \Omega$ *and* $t > 0$.

(A4) **Efficiency:** (a) *The security price process* $\boldsymbol{s}$ *is adapted to* $\boldsymbol{F}^m$ *and* (b) *the investor believes that, given the current market state, his private information gives no additional information about future market events; i.e., for any* $t > \tau$ *and* $A_t^m \in \mathcal{F}_t^m$, $P[A_t^m \mid \mathcal{F}_\tau] = P[A_t^m \mid \mathcal{F}_\tau^m]$.

This definition of partially complete markets (A3) is a natural generalization of complete markets that includes complete markets and non-existent markets as extreme cases. In one extreme, the market is complete in the usual sense if it is partially complete and $\boldsymbol{F}^m = \boldsymbol{F}$; in this case the efficiency assumption (A4) is trivially satisfied. In the other extreme, if the only security is the risk-free security, the market is still partially complete with $\mathcal{F}_t^m = \{\varnothing, \Omega\}$ for all t. The efficiency condition can be interpreted as saying that private information provides no information about security prices that is not already included in $\boldsymbol{F}^m$. As a consequence, we can completely describe the securities market by assigning probabilities and prices in the reduced model described by $\boldsymbol{F}^m$ (e.g., using the reduced tree of Figure 2 rather than the full tree of Figure 1).

Note that the efficiency condition requires market events to be independent of earlier private events but allows private uncertainties to be dependent on contemporaneous and previous market and private events. This is the case in the farmer example where low prices tend to accompany high yields and vice versa (see Figure 1).

3. Valuation

Given these restrictions on the investor's preferences and the form of incompleteness in the market, we can develop a procedure for evaluating projects without explicitly modeling the investor's trading decisions. The basic idea is to use subjective beliefs and preferences to determine effective certainty equivalents conditioned on the occurrence of a particular market state. We then determine the value of a project by determining the market value of a portfolio that matches these market-state-contingent effective certainty equivalents or, equivalently, using risk-neutral methods. In essence, in calculating

market-state-contingent certainty equivalents, we are projecting a problem with incomplete markets to an equivalent one where markets are complete. In a multiperiod setting, this projection is done recursively. We first focus on valuing a single project (i.e., a project with a fixed production plan) in Sections 3.1-3.3. We present a value additivity result in 3.4 and consider the selection of optimal production plans in Section 3.5.

The justification of this definition of "value" developed in this section will be provided by the Separation Theorem of the next section. There it is shown that optimal production plans maximize this definition of value and the amounts may be interpreted as a project's present certainty equivalent value or breakeven buying or selling price.

3.1. Effective Certainty Equivalent Projection

The key to the valuation procedure is to define a projection that maps from project cash flows in incomplete markets to an equivalent set of cash flows in the span of the complete market subspace. We can then use complete market methods to value spanned component. This projection is defined by replacing one-period's private risks with their *effective certainty equivalents* (as discussed informally in Section 2.1) and is done recursively, one period at a time. Let v_t denote a $\mathcal{F}_t$ measurable random variable, which later will represent the value of the project at the end of period t, after period t's uncertainties have been resolved. Let $\mathcal{F}_t^m \oplus \mathcal{F}_{t\text{-}1}$ denotes the smallest algebra containing both $\mathcal{F}_t^m$ and $\mathcal{F}_{t\text{-}1}$; describing what is known at time-(t-1) plus the time-t market information. The effective certainty equivalent mapping $\mathrm{ECE}_t[\text{-}]$ is defined by calculating certainty equivalents for period-t private risk using an exponential utility function with effective risk tolerance R_t:

$$\mathrm{ECE}_t[\, v_t \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t\text{-}1}] \;\equiv\; -R_t \ln(\mathrm{E}[\exp(-v_t/R_t) \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t\text{-}1}]) \tag{2}$$

$$R_t \;\equiv\; \sum_{\tau=t}^{T} \frac{\rho_\tau}{(1+r_f)^{\tau\text{-}t}} \;. \tag{3}$$

In (2), we condition on the time-(t-1) private state (contained in $\mathcal{F}_{t\text{-}1}$) and the occurrence of a particular market state at time t (contained in $\mathcal{F}_t^m$) and take expectations over the time-t private state. The operator changes each period as the effective risk tolerances change and income risks are spread over different numbers of remaining periods.

We can illustrate these definitions using the farmer example described in Figure 1. In period 2, the algebra $\mathcal{F}_t^m \oplus \mathcal{F}_{t\text{-}1}$corresponds to the state of information at the nodes representing the yield uncertainty where the period-2 price is known but the yield uncertainty has not yet been resolved. We would calculate effective certainty equivalents by taking expectations over the yield risks using an effective risk tolerance of $R_2 = \rho_2 = 40$. Focusing on the scenario where the early indication is positive and calculating effective certainty equivalents for the top yield node, we find an effective certainty equivalent for the high price scenario of $-R_2 \ln(.43\exp(-255/R_2) + .57\exp(-180/R_2)) = 198.11$. The interpretation of this effective certainty equivalent is that the investor is just indifferent between taking the gamble described by this yield node (.43 chance of receiving \$255 and .57 chance of receiving \$180) and receiving \$198.11 for certain. Similar calculations for the low price scenario give an effective certainty equivalent of 128.55.

This effective certainty equivalent operator thus decomposes the $\mathcal{F}_t$-measurable random variable v_t into two components

$$v_t = \mathrm{ECE}_t[\, v_t \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t\text{-}1}] + w_t \ .$$

The first component $\mathrm{ECE}_t[\, v_t \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t\text{-}1}]$ is a $\mathcal{F}_t^m \oplus \mathcal{F}_{t\text{-}1}$-measurable random variable that can be interpreted as the projection of v_t onto the marketed subspace. This component will be valued using contingent claims methods as described in the next two sections. The second component, is a residual term or a *windfall* $w_t \equiv v_t - \mathrm{ECE}_t[\, v_t \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t\text{-}1}]$ that has no value under the ECE operator, i.e., $\mathrm{ECE}_t[\, w_t \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t\text{-}1}] = 0$ (this is easily verified). The stream of these windfalls $\boldsymbol{w} = (w_0, w_1, ..., w_T)$ ($\boldsymbol{w} \in \boldsymbol{X}$) can be interpreted as the unhedgable private shocks to the project value.

Note that the ECE operator is non-linear in that, in general, $\mathrm{ECE}_t[av_t] \neq a\mathrm{ECE}_t[v_t]$. The ECE operator is also not additive in that, given two $\mathcal{F}_t$-measurable random variables v_t^{A} and v_t^{B}, in general, $\mathrm{ECE}_t[v_t^{\mathrm{A}} + v_t^{\mathrm{B}}] \neq \mathrm{ECE}_t[\, v_t^{\mathrm{A}}] + \mathrm{ECE}_t[\, v_t^{\mathrm{B}}]$. Additivity does however hold in the case where either v_t^{A} or v_t^{B} is $\mathcal{F}_t^m \oplus \mathcal{F}_{t\text{-}1}$-measurable. The projection both linear and additive in the limiting case of a risk-neutral investor (i.e., one with consumption risk tolerances ρ_t approaching ∞ for all t).

3.2. Valuation by Replication

After reducing the private risks by calculating effective certainty equivalents, we can value the market components using either replicating methods or, equivalently, using risk-neutral methods. We

consider replicating methods first. A *certainty-equivalent replicating trading strategy* for a project is a trading strategy that matches the project's effective certainty equivalent in each future market state. Given a project $\boldsymbol{p} \in \boldsymbol{X}$, since the project produces no income after time T, the project's final replicating portfolio $\boldsymbol{\theta}_{r,T}$ is equal to $\mathbf{0}$. In this case, and in general, the *time-t value* (v_t) of the project is defined as the market value of its certainty-equivalent replicating trading strategy plus any time-t project cash flows:

$$v_t \equiv p_t + \boldsymbol{\theta}_{r,t}\, \boldsymbol{s}_t \ . \tag{4}$$

Earlier certainty-equivalent replicating portfolios are defined recursively: given v_t, we find $\boldsymbol{\theta}_{r,t\text{-}1}$ (and hence $v_{t\text{-}1}$) by solving

$$\boldsymbol{\theta}_{r,t\text{-}1}\, \boldsymbol{s}_t = \mathrm{ECE}_t[\, v_t \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t\text{-}1}] \ . \tag{5}$$

If markets are partially complete we can always find trading strategies $\boldsymbol{\theta}_r$ that are adapted to $\boldsymbol{F}$ (and thus feasible) that satisfy (3) and this, together with the no arbitrage assumption, implies that the *value stream* $\boldsymbol{v} \equiv (v_0, v_1, ..., v_T)$ is unique and $\boldsymbol{F}$-adapted.

Proposition 1 (Valuation by Replication): *If markets are partially complete (A3), then, for any project* $\boldsymbol{p}$*, there exists a* $\boldsymbol{\theta}_r \in \boldsymbol{\Theta}$ *and a unique* $\boldsymbol{v} \in \boldsymbol{X}$*, satisfying equations (2) and (3).*

In the special case of complete markets or, more generally, for a project that lies in the span of the securities market (i.e., an $\boldsymbol{F}^m$-adapted project), the project values v_t are uniquely determined by the time-t market information and this valuation procedure reduces to the standard complete-markets replication procedure. In this case, the values and replicating trading strategies are independent of the investor's beliefs and preferences. If markets are incomplete, the values and replicating strategies depend on the investor's probabilities for private risks (i.e., the conditional probabilities $P(S_t \mid S_t^m, S_{t\text{-}1})$ used to calculate expectations in equation 4) and risk preferences (as captured by the consumption risk tolerances ρ_t), but are independent of his beliefs for market events (i.e., the probabilities $P(S_t^m)$) and time preferences (as captured by the utility weights k_t) as well as his initial wealth (e_0). Any dependence between market and private events is taken into account by calculating effective certainty equivalents conditioned on the market states.

We can illustrate this valuation procedure by considering the example of Figure 1. Focusing on the scenario where the early indication is positive, we earlier found effective certainty equivalents of 198.11

and 128.55 for the high and low price states, respectively. To find the certainty-equivalent replicating portfolio $\boldsymbol{\theta}_{r,2} = (\theta^0_{r,2}, \theta^1_{r,2})$, we solve equation (3) which in this context becomes:

$$(1.04)^2\ \theta^0_{r,2} \ + \ 2.00\ \theta^1_{r,2} \ = \ 198.11$$

$$(1.04)^2\ \theta^0_{r,2} \ + \ -3.00\ \theta^1_{r,2} \ = 128.55.$$

This gives $\boldsymbol{\theta}_{r,2} = (157.44, 13.91)$: given a positive early indication on yield, the certainty equivalent replicating portfolio contains 13.91 shares of the futures contract and 157.44 shares of the risk-free security. In this case, according to equation (2), the project is worth $v_1 = -5 + (157.44, 13.91) \times (1.04, 0.00) = 158.73$. Given a negative early indication on yield, similar calculations give a project value of 136.81.

Proceeding recursively, we then calculate an effective certainty equivalent for the period 1 using an effective risk tolerance $R_1 = 40 + 40/(1.04) = 78.46$. The period-1 effective certainty equivalent is $-R_1 \ln(.70\exp(-158.73/R_1) + .30\exp(-136.81/R_2)) = 151.49$. With no market uncertainty in this period, the replicating portfolio could contain only the risk-free bond or only the futures contract or a combination of the two (since both are risk-free); using only the risk-free bond, the replicating portfolio contains 145.66 = (151.49/1.04) bonds. Applying equation (4), we find a present value $v_0 = -130 + (145.66, 0) \times (1.00, 0.00) = 15.66$.

3.3. Risk-Neutral Valuation

Rather than explicitly constructing a certainty-equivalent replicating portfolio, we can also calculate project values using "risk-neutral" methods. When markets are arbitrage-free, there is a *risk-neutral measure* (or "equivalent martingale measure") P^* defined on $(\Omega, \mathcal{F})$ such that security prices may be calculated as expected future values discounted at the risk-free rate r_f, i.e., for any t and $\tau > t$,

$$\boldsymbol{s}_t \ = \ \frac{1}{(1+r_f)^{\tau-t}}\, \mathrm{E}^*[\ \boldsymbol{s}_\tau \,|\, \mathcal{F}_t] \tag{6}$$

where E* denotes expectations computed using P^* (see Harrison and Kreps 1979). In general, these risk-neutral probabilities will not be equal to the investor's (or any other investor's) probabilities and will be unique if and only if markets are complete. In our case, if we assume that markets are complete within $\boldsymbol{F}^m$, there will be unique risk-neutral probabilities for market events and we can determine unique risk-neutral expectations $\mathrm{E}^*[\ x \,|\, \mathcal{F}_t]$ for any $(\mathcal{F}^m_{t+1} \oplus \mathcal{F}_t)$-measurable random variable x. We can determine

these risk-neutral probabilities $P^*(S^m_{t+1} \mid S^m_t)$ working in a reduced tree focusing on the market risks (e.g., in the tree of Figure 2). We can then use the risk-neutral probabilities to construct a tree for the project with risk-neutral probabilities $P^*(S^m_{t+1} \mid S^m_t)$ in place of the corresponding subjective probabilities $P(S^m_{t+1} \mid S_t)$ for these market events. The "mixed tree" for the example is shown in Figure 3 with the risk-neutral probabilities marked with asterisks; the calculations for this example will be discussed shortly.[2]

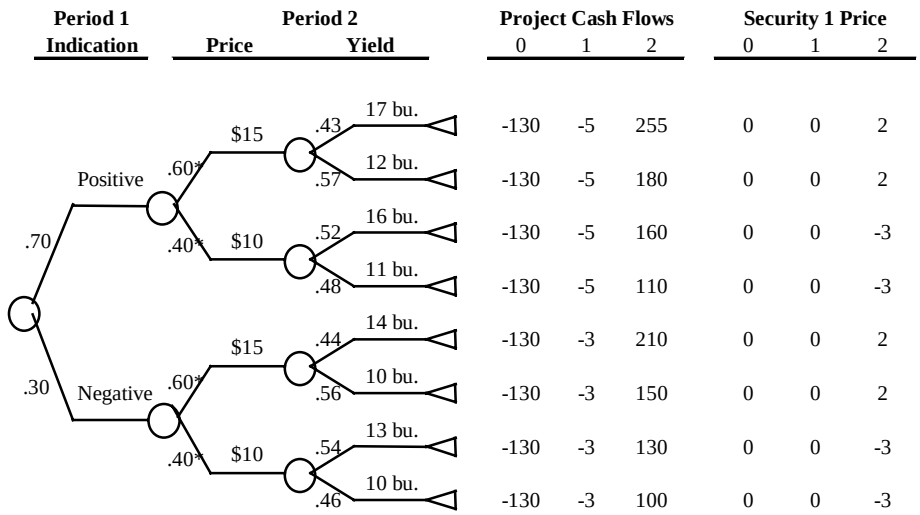

Figure 1: The farmer's problem.

To value a project using risk-neutral methods, we "roll back" this mixed tree, using subjective probabilities and risk tolerances to calculate effective certainty equivalents at nodes corresponding to private risks and using risk-neutral probabilities to calculate expected values at nodes corresponding to market risks. In a multiperiod framework, we again proceed recursively.

Proposition 2 (Risk-Neutral Valuation): *If markets are partially complete (A3), then we may compute project values recursively as follows. The time-T value of a project is given by $v_T = p_T$. For earlier times, the project values are given recursively by*

$$v_t = p_t + \frac{1}{(1+r_f)} \mathrm{E}^*[\mathrm{ECE}_{t+1}[\, v_{t+1} \mid \mathcal{F}^m_{t+1} \oplus \mathcal{F}_t] \mid \mathcal{F}_t] \tag{7}$$

where $\mathrm{E}^*[\,-\mid \mathcal{F}_t]$ *denotes expectations computed using the risk-neutral measure (defined in equation 6) and* $\mathrm{ECE}_{t+1}[\,-\mid \mathcal{F}^m_{t+1} \oplus \mathcal{F}_t]$ *denotes effective certainty equivalents (defined in equation 2).*

[2] To facilitate calculations in these trees, we adopt the convention of placing nodes for one period's market risk before the private risks for the same period. Thus a tree with market and private risk in each period, would have the market risk for period 1 and then the private risk for period 1, followed by the market risk for period 2 and then the private risk for period 2, and so on.

In the special case of complete markets, this procedure reduces to the standard risk-neutral valuation procedure, and, in the special case where the only security available is the risk-free security, this valuation procedure reduces to a special form of the recursive utility procedure developed in Kreps and Porteus (1978). In the case were the only security is the risk-free security, if we write the recursion of Proposition 2 in terms of "effective utilities", $\hat{U}_t \equiv -\exp(-v_t/R_t)$, we have a terminal utility $\hat{U}_T = -\exp(-p_T/R_T)$ and, for $t = 0, 1, \ldots, T-1$,

$$\hat{U}_{t\text{-}1} = \hat{u}_{t\text{-}1}(p_{t\text{-}1}, \mathrm{E}[\hat{U}_t \mid \mathcal{F}_{t\text{-}1}])$$

where $$\hat{u}_{t\text{-}1}(p, \hat{U}) = \exp(-p/R_{t\text{-}1})\, \hat{U}^{\,1/(1+r_f)\ (R_t/R_{t\text{-}1})}.$$

Thus, in this case, the valuation procedure reduces to a special case of the Kreps-Porteus recursive utility procedure with a specific form of terminal utility $\hat{U}_T$ and "basic utility function" (or "aggregator function") $\hat{u}_t$. The procedure in Proposition 2 can then be seen as a generalization of the Kreps-Porteus procedure that uses market information (in the form of risk-neutral probabilities) to value market risks. While one could consider other forms of utility and aggregator functions in a procedure like equation (7), the particular forms assumed here reflect the assumed additive-exponential utility function required for separation to hold. Project values computed using other forms may not correspond to present certainty equivalent values and may lead to project selections inconsistent with the solution to the investor's grand problem (1). (More on this in Section 5.1 below.)

We can illustrate the risk-neutral valuation procedure using the farmer example. First, we find the risk-neutral probabilities using equation (6). Letting p_1 denote the probability associated with the top branch in Figure 2 and p_2 the other probability, equation (6) becomes:

$$0.00 = (p_1\,(2.00) + p_2\,(-3.00))/(1.04)$$

$$1.04 = (p_1\,(1.04^2) + p_2\,(1.04^2))/(1.04).$$

Solving this, gives $p_1 = .60$ and $p_2 = .40$. Then we can calculate project values by rolling back the tree of Figure 3. Given a positive early indication, we have an effective certainty equivalent of 198.11 given a high price and an effective certainty equivalent of 128.55 given a low price; as described earlier. Given a positive early indication, the value in period 1 is then given by equation (7) as $v_1 = -5 + 1/1.04\,(.60\times198.11 + .40\times128.5) = 158.73$. Given a negative indication, we find a value of $v_1 = 136.81$. Rolling the tree further back and calculating effective certainty equivalents for period 1 and discounting, we find

an present value $v_0 = 15.66$. As required by Proposition 2, the values calculated using the risk-neutral approach are the same as those found earlier using replicating portfolios.

3.4. Value Additivity

The valuation procedure described so far considers a single project; what happens when we consider multiple projects that are undertaken simultaneously? With complete markets, we have the so-called "value additivity" principle that says the value of a portfolio of projects is equal to the sum of the values of each project. Moreover, we can hedge the portfolio of projects by hedging each project individually. To achieve similar results in this setting, we must place some restrictions on the projects in the portfolio. To formalize this "value additivity" result, let $\boldsymbol{v}(\boldsymbol{p})$ and $\boldsymbol{\theta}_r(\boldsymbol{p})$ denote the value stream and a replicating trading strategy for a project $\boldsymbol{p}$.

Proposition 3 (Value Additivity): *Assume markets are partially complete (A3) and efficient (A4). Given projects* $\boldsymbol{p}^A, \boldsymbol{p}^B \in \boldsymbol{X}$, *if* $\boldsymbol{p}^A, \boldsymbol{p}^B$ *are independent given* $\mathcal{F}_T^m$, *then*

a) $\boldsymbol{v}(\boldsymbol{p}^A + \boldsymbol{p}^B) = \boldsymbol{v}(\boldsymbol{p}^A) + \boldsymbol{v}(\boldsymbol{p}^B)$,

b) $\boldsymbol{\theta}_r(\boldsymbol{p}^A + \boldsymbol{p}^B) = \boldsymbol{\theta}_r(\boldsymbol{p}^A) + \boldsymbol{\theta}_r(\boldsymbol{p}^B)$, and.

c) $\boldsymbol{v}(\boldsymbol{p}^A)$ *and* $\boldsymbol{v}(\boldsymbol{p}^B)$ *are independent given* $\mathcal{F}_T^m$.

Informally, the independence condition of the proposition requires the project cash flows to have independent private risks, though the two projects may have common market risks. For example, two farms may both face the same price risks and have yield risks that are correlated, provided that yields are conditionally independent given prices. Part (c) of the proposition says that values satisfy this same independence condition. If markets are complete or both projects are spanned by the market, this independence condition is automatically satisfied (the project cash flows are determined by the market state) and this result reduces to the standard complete markets "value additivity" result. In the case of incomplete markets, the result follows from the fact that, given an exponential utility function, the certainty equivalent of the sum of two independent random variables is equal to the sum of the two certainty equivalents.

3.5. Project Selection and Sequential Decision Problems

Given partially complete markets, the valuation procedure developed in this section defines a unique present value (v_0) for any fixed project. If we have some flexibility in selecting or managing a

project and are given a family of projects $\boldsymbol{p}(\boldsymbol{\pi})$ where we can choose a production plan $\boldsymbol{\pi}$ from a set of available strategies $\boldsymbol{\Pi}$, we can determine a unique maximal project value and identify at least one optimal strategy $\boldsymbol{\pi}^*$. (This follows from our assumption that $\boldsymbol{p}(\boldsymbol{\pi})$ is a continuous function of $\boldsymbol{\pi}$ and $\boldsymbol{\Pi}$ is compact.) If we can decompose the production plans $\boldsymbol{\pi}$ into a sequence of decisions made in each period, we exploit the recursive nature of the valuation procedure to determine an optimal plan using dynamic programming techniques.

Proposition 4 (Sequential Decision Making): *A strategy* $\boldsymbol{\pi}^*$ *is optimal if and only if, for* $t = 0, 1, \ldots, T-1$,

$$v_t(\boldsymbol{\pi}^*) = \max_{\boldsymbol{\pi} \in \boldsymbol{\Pi}_t(\boldsymbol{\pi}^*)} \left\{ p_t + \frac{1}{(1+r_f)} \mathrm{E}^*[\mathrm{ECE}_{t+1}[\, v_{t+1}(\boldsymbol{\pi}) \mid \mathcal{F}^m_{t+1} \oplus \mathcal{F}_t \,] \mid \mathcal{F}_t] \right\}$$

where $\boldsymbol{\Pi}_t(\boldsymbol{\pi}^*) = \{ \boldsymbol{\pi} \in \boldsymbol{\Pi} \mid \pi_\tau = \pi^*_\tau \text{ for } \tau = 0, 1, \ldots, t \}$.

Here the set $\boldsymbol{\Pi}_t(\boldsymbol{\pi}^*)$ denotes the set of all strategies with the same decisions for the first t periods, and, Proposition 4 says a plan is optimal if and only if the actions chosen in each period are "optimal continuations" given the then-prevailing state and the decisions made up to that point. Thus we can adapt standard dynamic programming arguments for use in this framework. For example, Smith and McCardle (1998) adapt standard dynamic programming results on optimal stopping to determine the form of the optimal policy for abandoning an oil field.

4. Fisher Separation in Partially Complete Markets

In this section, we present an extension of the Fisher Separation Theorem applicable in partially complete markets. The result says that given the preference and market assumptions described in Section 2, an investor considering both productive opportunities and market opportunities for trading securities can decompose this grand problem into simpler production and portfolio-consumption problems that may be solved sequentially. In the production problem, the investor chooses among alternative projects to maximize their present values as given by the procedure of the previous section. In the portfolio-consumption problem, the investor chooses a security trading strategy to maximize his expected utility of consumption given that his initial wealth has been increased by the maximal project value. The optimal grand strategy is then given by composing the solutions to these two subproblems. We first discuss the

separation theorem in complete markets so we may develop the analogy between the complete and partially complete markets results.

4.1. Separation in Complete Markets

With complete markets, every project can be perfectly replicated by trading securities and project cash flows can literally be converted into their present certainty equivalent value. If the investor undertakes a project $\boldsymbol{p}$ and shorts its replicating trading strategy $\boldsymbol{\theta}_r$, all future project cash flows are exactly canceled and the net effect of the project is reduced to a lump-sum time-0 receipt of the project value v_0. If the investor undertakes a project and shorts its replicating trading strategy $\boldsymbol{\theta}_r$, then his *grand trading strategy*, $\boldsymbol{\theta}_g$ in (1), may be represented as $\boldsymbol{\theta}_s - \boldsymbol{\theta}_r$ where $\boldsymbol{\theta}_s$ represents the speculative securities investment, i.e. that part of the securities position that is not hedging project risks. Rewriting the grand problem using this representation of $\boldsymbol{\theta}_g$, the project cash flows are exactly canceled by the replicating trading strategy and the grand problem reduces to the *portfolio-consumption problem*:

$$\max_{\boldsymbol{\theta}_s \in \boldsymbol{\Theta}} \ \mathrm{E}[U(\boldsymbol{e} + \boldsymbol{v} + \boldsymbol{d}(\boldsymbol{\theta}_s))]$$

where $\boldsymbol{e} = (e_0, 0, 0, ..., 0)$ is the cash flow stream corresponding to a time-0 lump-sum receipt of the investor's endowed wealth (e_0), $\boldsymbol{v} = (v_0, 0, 0, ..., 0)$ is the cash flow stream corresponding to a time-0 lump-sum receipt of the project value (v_0), and $\boldsymbol{d}(\boldsymbol{\theta}_s)$ is the stream of proceeds generated by trading strategy $\boldsymbol{\theta}_s$.

Thus, when markets are complete we may decompose the grand problem (1) into production and portfolio-consumption problems that may be solved sequentially. In the production problem, the investor determines a production plan $\boldsymbol{\pi}^*$ that maximizes the project's present value. In the portfolio-consumption problem, the investor solves for a speculative trading strategy $\boldsymbol{\theta}_s^*$ that maximizes his expected utility of consumption given that his wealth has been increased by the maximal project value. The solution to the grand problem is to follow production plan $\boldsymbol{\pi}^*$ and the optimal grand trading strategy $\boldsymbol{\theta}_g^*$ is given by following the speculative trading strategy and shorting the optimal project's replicating strategy $\boldsymbol{\theta}_r^*$, i.e., $\boldsymbol{\theta}_g^* = \boldsymbol{\theta}_s^* - \boldsymbol{\theta}_r^*$.

4.2. Partially Complete Markets

When markets are partially complete, but not complete, if the investor undertakes a project and shorts its certainty-equivalent replicating trading strategy, project cash flows are generally not perfectly canceled. If the project's cash flows in some period exceed (or fall short) of the cash flows generated by the replicating trading strategy, the investor will rebalance his securities investments to spread this windfall (or shortfall) across future consumption. As discussed in 3.1, these windfalls w_t are defined as the private component of value $v_t - \mathrm{ECE}_t[\, v_t \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t\text{-}1}]$. Alternatively, noting the definition of the replicating portfolio in equation (5), the windfall as the difference between the value of the project at time-t and value of the replicating portfolio constructed in the previous period, $w_t = v_t - \boldsymbol{\theta}_{r,t\text{-}1}\, \boldsymbol{s}_t$. Or, equivalently, since $d_t(\boldsymbol{\theta}_r) = (\boldsymbol{\theta}_{r,t\text{-}1} - \boldsymbol{\theta}_{r,t})\, \boldsymbol{s}_t$ and $v_t = p_t + \boldsymbol{\theta}_{r,t}\boldsymbol{s}_t$, we can write the windfall as the difference between the cash flow generated by the project and that generated by the replicating portfolio, $w_t = p_t - d_t(\boldsymbol{\theta}_r)$.

Following the analogy with complete markets, suppose the investor undertakes a project $\boldsymbol{p}$ with present value v_0, shorts its certainty-equivalent replicating trading strategy $\boldsymbol{\theta}_r$, and adjusts his portfolio following a rebalancing trading strategy $\boldsymbol{\theta}_b$. We can then represent his grand trading strategy as $\boldsymbol{\theta}_g = \boldsymbol{\theta}_s - \boldsymbol{\theta}_r + \boldsymbol{\theta}_b$, where we can interpret the consumption strategy ($\boldsymbol{\theta}_s$) as a speculative investment in securities, the certainty-equivalent replicating portfolio ($\boldsymbol{\theta}_r$) as an *a priori* risk management strategy for reducing project risks before the uncertainties are resolved, and the rebalancing trading strategy ($\boldsymbol{\theta}_b$) as an *ex post* risk management strategy for bearing project risks after the uncertainties are resolved. Given our preference assumptions, this rebalancing takes a particularly simple form and involves only an adjustment in the holdings of the risk-free security, given by trading strategy $\boldsymbol{\theta}_{b,t} = (\theta_{b,t,0}, 0, ..., 0)$ where

$$\theta_{b,t,0} \equiv \frac{R_{t+1}}{(1+r_f)^{t+1}} \left(\sum_{\tau=1}^{t} \frac{w_\tau}{R_\tau} \right) \tag{10}$$

indicates the number of shares of the risk-free security held from period t to period t+1. Like the replicating trading strategy, this rebalancing trading strategy does not depend on the investor's probabilities for market events or his utility weights (k_t) and may be determined independently of the solution to the portfolio-consumption problem. The following lemma characterizes the investor's consumption stream using this representation of the grand trading strategy. Proof that (10) describes the optimal rebalancing strategy will follow shortly.

Lemma 2: *Let* $\mathbf{p}$ *be a project with present value* v_0*, certainty-equivalent replicating trading strategy* $\boldsymbol{\theta}_r$ *and rebalancing trading strategy* $\boldsymbol{\theta}_b$ *defined according to equation (10). Then, for any trading strategy* $\boldsymbol{\theta}_s$ *and* $\boldsymbol{\theta}_g = \boldsymbol{\theta}_s - \boldsymbol{\theta}_r + \boldsymbol{\theta}_b$*, we have the following:*

(a)
$$p_t + d_t(\boldsymbol{\theta}_g) = \begin{cases} v_0 + d_0(\boldsymbol{\theta}_s) & \textit{for } t = 0 \\ d_t(\boldsymbol{\theta}_s) + \rho_t \sum_{\tau=1}^{t} \frac{w_\tau}{R_\tau} & \textit{for } t > 0 \,. \end{cases}$$

(b)
$$u_0(e_0 + p_0 + d_0(\boldsymbol{\theta}_g)) = u_0(e_0 + v_0 + d_0(\boldsymbol{\theta}_s)), \qquad \textit{and}$$
$$\mathrm{E}[\, u_t(p_t + d_t(\boldsymbol{\theta}_g))\,] = \mathrm{E}[\, u_t(d_t(\boldsymbol{\theta}_s))\,], \qquad \textit{for } t > 0.$$

Here $p_t + d_t(\boldsymbol{\theta}_g)$ denotes the sum of the project cash flows and the cash flows generated by the grand trading strategy $\boldsymbol{\theta}_g$ and $d_t(\boldsymbol{\theta}_s)$ denotes the proceeds of the speculative trading strategy $\boldsymbol{\theta}_s$. The first part of the lemma says that if the investor manages project risks according to $\boldsymbol{\theta}_r$ and $\boldsymbol{\theta}_b$, the net effect of the project on consumption is reduced to a lump-sum receipt of the project value v_0 at time 0 plus a residual cash flow stream in which each period's windfall (w_t) is shared with future periods in proportion to each period's consumption risk tolerance. This risk sharing is analogous to Wilson's results on sharing risks among members of syndicate, as discussed in Section 2.1. The second part of the lemma shows that the expected utility in each period (expectations taken based on the information available at time 0) is the same if the investor undertakes the project and follows the grand trading strategy $\boldsymbol{\theta}_g$ or if he receives the project's value v_0 as lump-sum in period 0 and follows $\boldsymbol{\theta}_s$. This residual cash flow stream thus has no value to the investor.

When markets are complete, the residual cash flows are identically zero and we may ignore the project cash flows when solving for the optimal speculative investment $\boldsymbol{\theta}_s$ in the portfolio-consumption problem. When markets are incomplete the residual streams are not identically zero, but, provided assumptions (A1) – (A4) are satisfied, the project cash flows and private information can be safely ignored when solving the portfolio-consumption problem. This allows us to decompose the grand problem into simpler production and portfolio-consumption problems that may be solved sequentially.

Separation Theorem: *Suppose assumptions (A1)-(A4) are satisfied. Given a project* $\boldsymbol{p}(\boldsymbol{\pi})$, *let* $\boldsymbol{\pi}^*$ *denote a production plan that maximizes the present value of the project, let* v_0^* *be the maximal project value, and let* $\boldsymbol{\theta}_r^*$ *and* $\boldsymbol{\theta}_b^*$ *be certainty-equivalent replicating and rebalancing trading strategies for* $\boldsymbol{p}(\boldsymbol{\pi}^*)$. *Let* $\boldsymbol{\theta}_s^*$ *denote an* $\boldsymbol{F}^m$*-adapted trading strategy that solves*

$$\max_{\boldsymbol{\theta}_s \in \boldsymbol{\Theta}^m} \mathrm{E}\big[U(\boldsymbol{e} + \boldsymbol{v}^* + \boldsymbol{d}(\boldsymbol{\theta}_s))\big], \qquad \text{(\textit{Portfolio-consumption Problem})}$$

where $\boldsymbol{e} = (e_0, 0, 0, ..., 0)$ and $\boldsymbol{v}^* = (v_0^*, 0, 0, ..., 0)$. *Then* $\boldsymbol{\pi}^*$ *and* $\boldsymbol{\theta}_g^* = \boldsymbol{\theta}_s^* - \boldsymbol{\theta}_r^* + \boldsymbol{\theta}_b^*$ *is a solution to*

$$\max_{\boldsymbol{\pi} \in \boldsymbol{\Pi},\, \boldsymbol{\theta}_g \in \boldsymbol{\Theta}} \mathrm{E}\big[U(\boldsymbol{e} + \boldsymbol{p}(\boldsymbol{\pi}) + \boldsymbol{d}(\boldsymbol{\theta}_g))\big] \quad . \qquad \text{(\textit{Grand Problem})}$$

In the special case of complete markets, this separation theorem reduces to the complete markets result and the production decisions are entirely independent of the investor's subjective beliefs and preferences. With incomplete markets, if assumptions (A1)-(A4) are satisfied, the production decisions are independent of the investor's beliefs about market events, his time preferences (i.e., the utility weights k_t) and initial wealth (e_0), but depend on his beliefs about private events and his risk preferences (i.e., the consumption risk tolerances ρ_t). The solution to the portfolio-consumption problem depends on the investor's beliefs about the market states as well as his time and risk preferences and initial wealth, but requires no private information beyond the project value v_0^*. After increasing his wealth by v_0^*, the investor can focus exclusively on the securities investments and he can solve a simpler problem set in $\boldsymbol{F}^m$. For example in the farmer's problem, this means that the farmer can determine the optimal speculative trading strategy ($\boldsymbol{\theta}_s^*$) working in the tree of Figure 2 rather than that of Figure 1.

The Separation Theorem says that the production plans ($\boldsymbol{\pi}$) that are optimal in the grand problem (1) are those that maximize this definition of value. Since the investor is indifferent between two projects with the same value, we can also interpret these values as *present certainty-equivalent values* (or *breakeven selling prices*): the investor is indifferent between undertaking the project and receiving its value v_0 as a lump-sum for certain at time 0. The values can also be interpreted as *breakeven buying*

prices in that the investor is willing to pay up to v_0 to obtain the rights to the project.[3] In this sense, the valuation procedure plays a role analogous to the "NPV rule" in the deterministic framework considered by Fisher.

We can illustrate the separation result using the farmer example. Because the optimal speculative trading strategy ($\boldsymbol{\theta}_s^*$) is adapted to the market filtration of Figure 2, it does not vary depending on the early indication revealed in period 1. Specifically, we find that $\boldsymbol{\theta}_{s,0}^*$ = (76.68, 0) and $\boldsymbol{\theta}_{s,1}^*$ = (37.68,–4.20), where the first entry denotes the number of shares of the risk-free security and the second the position in the forward contract.[4] We may then derive the optimal grand trading strategy using the result of the Separation Theorem. For example, if we have a positive early indication in period 1 the farmer should hold the portfolio $\boldsymbol{\theta}_{g,0}^* = \boldsymbol{\theta}_{s,0}^* - \boldsymbol{\theta}_{r,0}^* + \boldsymbol{\theta}_{b,0}^*$ = (37.68, –4.20) – (–157.44, 13.91) + (3.41, 0) = (-116.33, –18.12), where the replicating portfolio is as given in Section 3.2 and the amount of the risk-free-security in the rebalancing portfolio is given by equation (10). Here we see that the farmer takes a speculative short position (–4.20 shares) in the forward contract, reflecting his relatively pessimistic view of the future prices (his probability for the high price is .47 compared to the risk-neutral probability of .60). This, when coupled with his hedging activities (short 13.91 shares) leads to a much larger short position (–18.12 shares). The decomposition of the grand problem of the Separation Theorem thus provides some insight into the structure of the optimal trading strategies as well as simplifying the computation of these strategies.

5. Extensions

In this section, we consider the implications of relaxing the assumptions made in deriving the separation theorem. We first consider relaxing the preference assumptions (A1 and A2) and then consider relaxing the market assumptions (A3 and A4). We also consider the possibility of incorporating non-negativity constraints on consumption in the grand problem.

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[3] Breakeven buying and selling prices are generally not equal, but here the equality follows from the absence of wealth effects. To see this, suppose project $\boldsymbol{p}$ has value v_0. Subtracting v_0 from the projects period-0 cash flow results in a project with value 0. Thus the investor is just indifferent to paying v_0 for $\boldsymbol{p}$ and forgoing the project.

[4] Note that since both securities are risk-free in period 1, without loss of optimality, we have arbitrarily taken the forward contract position in period-0 to be to be zero.

5.1. Preference Assumptions

While the assumption of time-additive utilities (A1) is fairly standard, the assumption of constant absolute risk aversion for period consumption (A2) is perhaps troublesome as it is inconsistent with the standard economic intuition that suggests risk aversion should be decreasing in wealth. We first examine the necessity of these conditions and then state a useful approximation result applicable when preferences do not exhibit constant absolute risk aversion.

Necessity. With incomplete markets, the investor will wind up holding residual risks for some projects. In order to achieve separation, the investor's preferences for these residual risks must be independent of the distribution of securities payoffs. For this to be the case, additivity is required to ensure that the investor's preferences for residual risks in one period are independent of the distribution of securities payoffs in other periods (see Fishburn 1970; Theorem 11.1). Constant absolute risk aversion is required to ensure that the market-state-contingent valuations in one period are independent of the securities payoffs in the same period. If risk aversion were varying with consumption levels, the market-state-contingent valuations would depend on the securities positions, which would, in turn, depend on the probabilities for the market states. Thus, if we allow incompleteness in all periods, both additivity and constant absolute risk aversion are necessary in order to be able to evaluate project and securities investments independently.

If, as in our model, we do not have any uncertainty in period 0, we can generalize these conditions slightly. In this case, the investors' preferences for consumption need only satisfy additive-independence and constant absolute risk aversion for future periods, conditioned on the period-0 consumption. Rather than the strictly additive exponential form described in Section 2.1, we can then have utility functions of the form:

$$U(c_0, c_1, \ldots, c_T) = a(c_0) + b(c_0)\left(\sum_{t=1}^{T} k_t\, u_t(c_t)\right) \tag{11}$$

where $b(c_0) > 0$ and, for $t > 1$, the period utilities u_t are exponential as before. Note that the procedures and results of Section 3 and the results of Lemma 2 are independent of the period-0 consumption risk

tolerance (ρ_0) and utility weight (k_0) and are unaffected by this change. The statement of the Separation Theorem is unchanged and its proof needs only slight modifications for this generalization.[5]

Approximation. Given that constant absolute risk aversion is necessary for separation, one might interpret the main message of the paper to be negative in that separation holds if and only if we make very restrictive assumptions about the investor's preferences. Alternatively, and more constructively, we can interpret the valuation procedure as a useful approximation. To formalize this sense of approximation, we maintain the assumption of additivity (A1) and relax the assumption of constant absolute risk aversion (A2). Given a fixed project $\boldsymbol{p} \in \boldsymbol{X}$, we can write the investor's problem recursively, as

$$U_t(e_t, \boldsymbol{p}) \equiv \max_{\boldsymbol{\theta}_t} \left\{ k_t u_t(e_t + p_t - \boldsymbol{\theta}_t \boldsymbol{s}_t) + \mathrm{E}[U_{t+1}(\boldsymbol{\theta}_t \boldsymbol{s}_{t+1}, \boldsymbol{p}) \mid \mathcal{F}_t] \right\} \tag{12}$$

for $t = 0$ to T-1 with the terminal case being $U_T(e_T, \boldsymbol{p}) = k_T u_T(e_T + p_T)$. Here e_t denotes the investor's wealth at the beginning of period t. The investor's overall time-0 utility defined is given as $U_0(e_0, \boldsymbol{p})$ and the optimal grand trading strategy $\boldsymbol{\theta}_g^*$ is given as the sequence of solutions $(\boldsymbol{\theta}_0^*, \boldsymbol{\theta}_1^*, \ldots, \boldsymbol{\theta}_T^*)$ to the optimization problem in (12), starting from wealth e_0 at time 0. We will assume that such a trading strategy exists.

In this setting, we can define the time-t certainty equivalent value of a project v_t as in section 4.2 as the lump-sum amount v_t such that the investor is indifferent between receiving v_t as income in period t (for certain) and continuing the project and receiving its uncertain income stream. Here, unlike the case where we assume constant absolute risk aversion, these project values will depend on the investor's wealth at time t, and can be defined formally as the $v_t(e_t, \boldsymbol{p})$ such that $U_t(e_t + v_t(e_t, \boldsymbol{p}), \boldsymbol{0}) = U_t(e_t, \boldsymbol{p})$ where $\boldsymbol{0}$ is a project paying 0 at all times and in all states. While these values are in general difficult to

[5] It is interesting to note that these preference conditions are essentially equivalent to those required for the Kreps-Porteus recursive utility procedure to represent induced preferences for income. In the case where the only security is the risk-free security, the grand problem (1) reduces to the "consumption-savings problem" studied in Kreps and Porteus (1979b). There they show that, in the two period case, a necessary and sufficient condition for induced preferences for income to be represented in their recursive utility framework is that the investor's utility function for consumption be of the form $U(c_0, c_1) = f(c_0) + g(c_0)\, h(c_0(1+r_f) + c_1)$ where $g(c_0) > 0$ and f, g, and h may depend on r_f. If we further require the investor's preferences for consumption to be independent of r_f, we find that the utility function must be of the form of equation (11), with a(-) and b(-) independent of r_f. Thus, the valuation procedure developed here can be viewed as an extension of the Kreps-Porteus procedure in the one case where the Kreps-Porteus procedure accurately represents induced preferences for income.

compute, if we can place upper and lower bounds on the investor's risk tolerances, $\rho_t(c_t) \equiv -u_t'(c_t)/u_t''(c_t)$, we can use the valuation procedure of Section 3 to generate upper and lower bounds on these values.

Proposition 5: *Suppose assumptions (A1), (A3) and (A4) are satisfied and the investor's period utility functions u_t are strictly risk-averse and twice continuously differentiable. Let $\boldsymbol{\theta}_g^*$ denote an optimal grand trading strategy for project $\boldsymbol{p} \in \boldsymbol{X}$, and let v_t denote the time-t value of $\boldsymbol{p}$ determined using the investor's true preferences assuming the investor follows the optimal trading strategy. Let $\overline{\rho}_t \geq \max\{\rho_t(c_t) : c_t = p_t + d_t(\boldsymbol{\theta}_g^*)\}$, $\underline{\rho}_t \leq \min\{\rho_t(c_t) : c_t = p_t + d_t(\boldsymbol{\theta}_g^*)\}$, and let $\overline{v}_t$ and $\underline{v}_t$ denote the time-t values of $\boldsymbol{p}$ computed assuming constant consumption risk tolerances $\overline{\rho}_\tau$ and $\underline{\rho}_\tau$ (respectively) for all $\tau \geq t$. Then $\underline{v}_t \leq v_t \leq \overline{v}_t$.*

Thus the accuracy of values computed assuming a constant risk aversion depends on how much the investor's risk tolerance varies over the range of possible consumption levels and how much of the project's risks can be hedged by trading. If the project can be perfectly hedged, the valuation procedure reduces to the standard contingent claims procedure and the values are independent of the investor's risk tolerance. In this case, the upper and lower bounds are equal. If there are private (unhedgable) risks, then the width of these bounds depends on the magnitude of the private risks and the change in risk tolerance over the range of possible consumption levels. The range of possible consumption levels depends on both the magnitude of the private risks and the magnitude of the speculative risks the investor takes in the securities market. If the investor takes speculative positions that lead to large variations in consumption and risk tolerances, then the bounds on value may be quite wide. If, however, these speculative positions are smaller, for projects without large private risks, it may be reasonable to assume that risk aversion is approximately constant. In these cases, the valuation procedure of Section 3 will generate approximate values and "approximately optimal" production plans.

5.2. Market Restrictions

We now consider the implications of relaxing the market assumptions required for the separation result. First, suppose that the partial completeness (A3) condition fails and the securities market is not complete within the market filtration $\boldsymbol{F}^m$. In this case, for some projects there will be no certainty-equivalent replicating portfolios and we cannot determine project values using the replication method. Though there will still be risk-neutral probabilities for the market states, these risk-neutral probabilities will no longer be unique and the risk-neutral valuation procedure may not generate unique values.

Though we may uniquely define project values as their present certainty-equivalent value, even if the preference assumptions are satisfied, these values may depend on the investor's subjective probabilities for market states. (This can be verified by deleting a security in the numerical example and calculating present certainty equivalent values with different probabilities for the market states.) Thus, if the securities market is not partially complete, at least this aspect of the separation result fails.

Second, suppose that the efficiency conditions fails, so there is a filtration $\boldsymbol{F}^m$ (perhaps the null filtration with $\mathcal{F}_t^m = \{\varnothing, \Omega\}$ for all t) that is spanned by existing securities, but there exist securities whose prices are not independent of the private information. While we can determine unique project values using the valuation procedure of Section 3, the market-state-contingent effective certainty equivalents would not properly take into account the dependence between securities and project cash flows. The values calculated would (in general) no longer be equal to the project's present certainty equivalent value. Consequently, production plans selected to maximize this definition of value may no longer be optimal in the grand problem (1). Moreover, even if one were to somehow determine the present certainty equivalent values, because the private states give information about the securities prices, we could not restrict our attention to the market filtration $\boldsymbol{F}^m$ when solving the portfolio-consumption problem.

5.3. Non-Negativity Constraints

Finally, one might also hope to impose non-negativity constraints on consumption in the grand problem (1). While such constraints pose no problems with complete markets, if we impose these constraints with incomplete markets, we can no longer decompose the grand problem into production and portfolio-consumption problems that can be solved separately. With complete markets, since project cash flows are exactly canceled by the replicating portfolio, a non-negativity constraint on consumption in the grand problem translates to an identical non-negativity constraint in the portfolio-consumption problem. When markets are incomplete, the project cash flows are generally not perfectly canceled by the certainty-equivalent replicating portfolio and, if we were to impose a non-negativity constraint in the grand problem, the feasible cash flow streams in the portfolio-consumption problem would depend on the project cash flows. Thus, the value of the project can no longer be summarized by a single lump-sum increase in wealth and we could no longer focus exclusively on the market filtration ($\boldsymbol{F}^m$) when solving the portfolio-consumption problem. Moreover, if the non-negativity constraints are binding in the grand

problem, the present certainty-equivalent value of a project may depend on the investor's subjective probabilities for the market states.

6. Separation and Valuation for Firms

As discussed in the introduction, in the deterministic and complete market cases, the Fisher Separation Theorem provides a justification for both the "NPV rule" for evaluating projects and the separation of ownership and management. Throughout this paper, our focus has been on evaluating projects from an individual's perspective and we have developed the analog of the NPV rule for partially complete markets. We now briefly consider the separation and valuation issues in the case of a project owned by multiple investors, as in a partnership, firm, or a syndicate.

6.1 Separation and Valuation in Syndicates

When markets are complete, investors with diverse beliefs and preferences may cooperate on projects and, with no loss, may delegate project management to a manager who need not know anything about the investors' beliefs and preferences. The present certainty equivalent value of a project is unambiguously defined and the manager's job is to maximize this value. When markets are partially complete and efficient and the investor satisfies the necessary preference restrictions, an individual investor may still delegate production decisions to a manager and separately solve his own portfolio-consumption problem. In order to make production decisions that are optimal for the investor, the manager needs to know the investor's risk tolerances and his probabilities for private events (i.e., the conditional probabilities $P(S_t \mid S_t^m, S_{t-1})$ used in the valuation procedure of Section 3). When investors with diverse beliefs and preferences cooperate and share the risks of a project, they need to develop mechanisms for making production decisions.

What might these mechanisms look like? Wilson's "theory of syndicates" (Wilson 1968) provides some results along these lines. Wilson examines a single-period model and considers a group of investors (the syndicate) sharing some risky project. If the investors all have exponential utilities, Wilson shows that in a Pareto optimal sharing arrangement the investors would place bets with each other on the state of the world (only if they disagree on probabilities) and own shares in the risky project in proportion to their risk tolerances; they may also make deterministic side payments. The syndicate, as a Pareto efficient group, would then make decisions as if it were a single investor with an exponential

utility function with a risk tolerance equal to the sum of the individual investors' risk tolerances and probabilities equal to the geometric mean of their probabilities, weighted in proportion to their shares. Given this sharing system, all investors will agree on the choice of projects and production plans, unanimously supporting that which maximizes the expected utility given by using the syndicate's probabilities and utilities.

Wilson's results generalize directly to our setting. To formalize these results, let us consider a group of I investors, labeled by superscripts $i = 1, 2, \ldots, I$. Each investor's beliefs are captured by a probability measure P^i defined on $(\Omega, \mathcal{F})$, where we assume (as in Section 1) that each measure is strictly positive in that $P^i(A) > 0$ for all non-null events $A \in \mathcal{F}$. Thus, though we allow the possibility of differences in beliefs, we assume that the investors agree on what is possible. We further assume that all investors receive the same information (i.e., they share the same filtration $\boldsymbol{F}$) and have access to the same securities markets and trade at common prices, maintaining the market assumptions of Section 1. Then applying Wilson's result recursively in the partially complete market setting, we find the following.

Proposition 6 (Syndicates in Partially Complete Markets): *Suppose that each investor in the syndicate satisfies assumptions (A1) and (A2) with utility weights k_t^i and consumption risk tolerances ρ_t^i. Further suppose that the market is partially complete (A3) and that each investor believes that the market is efficient (A4). Then, in a Pareto optimal sharing arrangement:*

(a) *The syndicate will select production plans using the valuation procedure of Section 3 with effective risk tolerances $R_t^0 \equiv \sum_{i=1}^{I} R_t^i$ and probabilities*

$$P_t^0(S_t \mid S_t^m, S_{t-1}) \equiv \prod_{i=1}^{I} P(S_t \mid S_t^m, S_{t-1})^{R_t^i/R_t^0} \quad .$$

(b) *Each investor receives a share of the period-t project windfalls (w_t) that is proportional to their effective risk tolerance (i.e., each investor receives $w_t(R_t^i/R_t^0)$), for each $t > 0$.*

(c) *All investors unanimously support the strategies selected using this valuation procedure.*

Thus, considering an investment decision made by a firm, we can use the valuation procedure of Section 3 with aggregate beliefs and preferences in the same way as Wilson's model except here a manager need only query investors about their probabilities for private events ($P^i(S_t \mid S_t^m, S_{t-1})$). Like the individual production decisions, the syndicate's decisions are independent of the investors' beliefs about market events and their time preferences. Note that it is the investors' effective risk tolerances (R_t^i) that

determine their preferences for income in period t and, hence, determines their weights in the probability weighting scheme of part (a) and their shares in part (b). If all of the investors have common probabilities for the private events, then the syndicate's probabilities will be equal to these common probabilities. If the investors disagree on these probabilities, then the syndicate's probabilities, by virtue of taking geometric means, will sum to less than one; this has no economic consequences provided we take account of this fact when effective certainty equivalents.[6] In the multiperiod setting, the weights used to aggregate the probabilities may vary by period as the investors' consumption risk tolerances and share holdings vary. Consequently, the syndicate's probabilities for an event may change over time in manner inconsistent with Bayes' rule.

Also note that the sharing rule of part (b) is not unique in being Pareto efficient. Given their ability to trade securities, the investors will be indifferent among all income streams that generate $w_t(R_t^i/R_t^0)$ in period-t value, regardless of the precise timing of the cash flows: the firm could add the cash flows generated by any $\boldsymbol{F}$-adapted trading strategy $\boldsymbol{\theta}$ to the investor's dividends and the investor could undo this modification by subtracting their share of $\boldsymbol{\theta}$ from his own strategy. (This is the logic of the famous Modigliani-Miller "irrelevance" theorem; see, e.g., DeMarzo 1988.) The particular payment scheme of part (b) distributes project cash flows to the investors as soon as possible (i.e., as soon as the uncertainties are resolved) and does not require the investor to adjust his position in any security other than the risk-free securities. Alternatively, the syndicate could spread the period-t windfalls over all subsequent periods by paying each investor $w_t(\rho_\tau^i/R_t^0)$ in each period $\tau \geq t$ (as in Lemma 2a). In this case, the period-t value of the stream associated with period-t windfall would still be $w_t(R_t^i/R_t^0)$ and the investors would not have to adjust any securities positions; each investor would simply consume the cash flows as they are received.

[6] In calculating these effective certainty equivalents, we must note that the expected utility of the certain amount is less than the utility of that amount; this leads to a renormalization of the probabilities when calculating effective certainty equivalents. For example, suppose the farmer in the example were to share his crop with some other investor with the same period risk tolerances but who thought the probability of a high yield given high prices and a positive early indication is .65 instead of .43 as assumed by the farmer. As a group, the two of them should take the probability of this event to be $(.43^{0.5})(.65^{0.5}) = .5287$ and its complement to be $(.57^{0.5})(.35^{0.5}) = .4400$. The effective certainty equivalent for this node is then given by $-R_2^0 \ln\big((.5287\exp(-255/R_2) + .4400\exp(-180/R_2^0))/(.5287 + .4400)\big) = 212.01$ where $R_2^0 = R_2^1 + R_2^2 = 40 + 40 = 80$.

6.2 Discussion

Thus, given the assumptions required for separation to hold in incomplete markets, investors can cooperate on projects and delegate project management to managers (or firms) in much the same way as in complete markets. As in complete markets, the manager's job is to choose projects to maximize the market value of the project, but, to the extent that markets are incomplete, there is some subjectivity in this definition of value. In this setting, managers must poll the investors about their risk preferences and their beliefs concerning the private uncertainties, aggregating them as in Proposition 6.

For a firm with many investors (for example, a large publicly held firm), the aggregation of investor beliefs and preferences required by Proposition 6 would seem to be impractical. Taking Wilson's arguments to the limit and summing risk tolerances over all shareholders, one could argue that a large company should be essentially risk-neutral for all but the largest investments. In so far as beliefs are concerned, most investors would not have much information about the private opportunities facing the company and, if asked for probabilities, would defer to management's judgment. In this case, if management uses the valuation procedure of Section 3 to evaluate projects, for "small" projects, it would seem appropriate for them to adopt risk-neutral preferences and attempt to estimate aggregate probabilities for private events. In this case, the ECE operator of Section 3.1 is replaced by the expectations operator and by construction, the windfalls w_t are uncorrelated with all securities prices (since $E[\, v_t \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t\text{-}1}] = 0$ for all t). The valuation procedure of 3.3 reduces to standard dynamic programming, taking expectations at every node and discounting at the risk-free rate. The firm would be risk-neutral with respect to private risks, but, because it uses risk-neutral probabilities to evaluate market risks, the firm would appear to be risk-averse with respect to market risks.

It may be instructive to compare this valuation procedure with those discussed in the real options literature. Dixit and Pindyck (1994; pp. 120-121) propose using contingent claims methods (e.g., dynamic programming with risk-adjusted probabilities or replicating portfolio methods) in cases where the project uncertainties are spanned by the market; cash flows are discounted at the risk-free rate. When spanning does not hold, they suggest using dynamic programming with an "arbitrary discount rate" (pg. 148) that "can simply reflect the decision maker's subjective valuation of risk" (pg. 121). The valuation method developed here is helpful in the middle ground where some risks may be hedged and others cannot. Here the risk-adjustments are incorporated by risk-adjusting the probabilities for market risks

(i.e., using risk-neutral probabilities instead of true probabilities) and, if the stakes are sufficiently large, using the investor's or investors' risk tolerances to assign risk premiums for private risks; cash flows are always discounted at the risk-free rate. Any correlation between market and private risks is captured by explicitly modeling the dependence between them and using risk-neutral probabilities for the market risks.

7. Summary

In summary, we see this paper as making two contributions. First, we have identified conditions that allow investors to separate production and portfolio-consumption decisions with incomplete markets. In as much as markets in the real world are incomplete and production and portfolio-consumption problems are typically solved separately, it is important to understand when and how separation may be achieved without loss. Second, we have developed a new procedure for valuing real projects that cannot be replicated by trading existing securities. This new procedure extends the complete-markets contingent claims valuation methods and provides a simple method for computing project values and management strategies that are consistent with market prices for existing securities and investor's or investors' subjective beliefs and preferences.

Appendix: Proofs

Lemma 1: We first consider the case of a fixed project $\boldsymbol{p}$ and show that there exists an optimal trading strategy $\boldsymbol{\theta}_g$. Let

$$\mathrm{EU}(\boldsymbol{\theta}_g) \equiv \mathrm{E}[\ U(\boldsymbol{p} + \boldsymbol{d}(\boldsymbol{\theta}_g))\ ] = \sum_{\omega} \sum_{t=0}^{T} - \mathrm{P}(\{\omega\})\ k_t\ \exp(-(p_t(\omega) + d_t(\omega; \boldsymbol{\theta}_g))/\rho_t\ ).$$

For any feasible trading strategy $\boldsymbol{\theta}_g$ such that $\boldsymbol{d}(\boldsymbol{\theta}_g) \neq \boldsymbol{0}$, there exist at least one time-state (t,ω) such that $d_t(\omega; \boldsymbol{\theta}_g) < 0$ and another such that $d_t(\omega; \boldsymbol{\theta}_g) > 0$, otherwise either $\boldsymbol{\theta}_g$ or $-\boldsymbol{\theta}_g$ would be an arbitrage opportunity. Considering trading strategies $\kappa\boldsymbol{\theta}_g$ with proceeds $\kappa\boldsymbol{d}(\boldsymbol{\theta}_g)$, we find that $\lim_{\kappa\to\pm\infty} \mathrm{EU}(\kappa\boldsymbol{\theta}_g) = -\infty$, since in each time-state (t,ω),

$$\lim_{\kappa\to\infty} \exp(-(p_t(\omega) + d_t(\omega; \kappa\boldsymbol{\theta}_g))/\rho_t) = \begin{cases} 0 & \text{if } d_t(\omega; \boldsymbol{\theta}_g) > 0 \\ \infty & \text{if } d_t(\omega; \boldsymbol{\theta}_g) < 0 \end{cases}.$$

Thus, since $\mathrm{EU}(\boldsymbol{\theta}_g)$ is a concave function of $\boldsymbol{\theta}_g$ (having inherited the concavity of U) that asymptotically approaches $-\infty$ in all feasible directions, the optimization problem $\max_{\boldsymbol{\theta}_g} \mathrm{EU}(\boldsymbol{\theta}_g)$ has an interior solution. Given a choice of production plans, the existence of a $\boldsymbol{\pi}^*$ that solves (1) then follows from our

assumption that $\boldsymbol{p}(\boldsymbol{\pi})$ is a continuous function of $\boldsymbol{\pi}$ where $\boldsymbol{\pi}$ is restricted to some compact set (see, e.g., Royden 1968, pg. 161). ///

Proposition 1: We need to show that for all t, $\boldsymbol{\theta}_{r,t}$ and v_t exist and are $\mathcal{F}_t$-measurable and v_t is unique. We establish this result using induction. At time $t = T$, the result is trivial. Assuming that result holds for time t, we need to show that the same properties hold at t-1. The conditional expectations in (4) ensure that the right side of (3) is a $(\mathcal{F}_t^m \oplus \mathcal{F}_{t-1})$-measurable random variable for all $t > 0$. The assumption of partially complete markets (A3) then implies the existence of a $\mathcal{F}_{t-1}$-measurable $\boldsymbol{\theta}_{r,t-1}$ that solves (3). Since p_{t-1}, $\boldsymbol{\theta}_{r,t-1}$, and $\boldsymbol{s}_{t-1}$ are all $\mathcal{F}_{t-1}$-measurable, $v_{t-1} \equiv p_{t-1} + \boldsymbol{\theta}_{r,t-1}\, \boldsymbol{s}_{t-1}$ is also $\mathcal{F}_{t-1}$-measurable. The uniqueness of v_{t-1} follows from our no-arbitrage assumption: though $\boldsymbol{\theta}_{r,t-1}$ need not be unique, if $\boldsymbol{\theta}^1_{r,t-1}\, \boldsymbol{s}_{t-1} \neq \boldsymbol{\theta}^2_{r,t-1}\, \boldsymbol{s}_{t-1}$ for two $\boldsymbol{\theta}^1_{r,t-1}$ and $\boldsymbol{\theta}^2_{r,t-1}$ that both satisfy equation (3), then either a trading strategy $\boldsymbol{\theta}^*$ taking $\boldsymbol{\theta}^*_{t-1} = \boldsymbol{\theta}^1_{r,t-1} - \boldsymbol{\theta}^2_{r,t-1}$ for this t and $\boldsymbol{\theta}^*_{\tau-1} = \mathbf{0}$ for all other times would be an arbitrage opportunity, or else $-\boldsymbol{\theta}^*$ would be.///

Proposition 2: We establish this result using induction. At time $t = T$, since the project generates a certain cash flow p_T and no future cash flows, $\boldsymbol{\theta}_{r,T} = \mathbf{0}$ and $v_T = p_T$. Assuming that equation (7) holds for time t, we show that it holds for time t-1 by establishing the following sequence of equalities:

$$\begin{aligned} v_{t-1} &= p_{t-1} + \boldsymbol{\theta}_{r,t-1}\, \boldsymbol{s}_{t-1} \\ &= p_{t-1} + \frac{1}{(1+r_f)} \mathrm{E}^*[\, \boldsymbol{\theta}_{r,t-1}\, \boldsymbol{s}_t \mid \mathcal{F}_{t-1}] \\ &= p_{t-1} + \frac{1}{(1+r_f)} \mathrm{E}^*[\mathrm{ECE}_t[\, p_t + \boldsymbol{\theta}_{r,t}\, \boldsymbol{s}_t \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t-1}] \mid \mathcal{F}_{t-1}] \end{aligned}$$

The first equality is simply the definition of v_{t-1}. The second equality follows from equation (6) defining the risk-neutral measure and the third equality follows from our definition of the certainty-equivalent replicating trading strategy. Equation (7) then follows from the induction hypothesis. ///

Proposition 3: We use induction to show that, for any t, (i) $v_t(\boldsymbol{p}^{\mathrm{A}} + \boldsymbol{p}^{\mathrm{B}}) = v_t(\boldsymbol{p}^{\mathrm{A}}) + v_t(\boldsymbol{p}^{\mathrm{B}})$, (ii) $\boldsymbol{\theta}_{r,t}(\boldsymbol{p}^{\mathrm{A}} + \boldsymbol{p}^{\mathrm{B}}) = \boldsymbol{\theta}_{r,t}(\boldsymbol{p}^{\mathrm{A}}) + \boldsymbol{\theta}_{r,t}(\boldsymbol{p}^{\mathrm{B}})$, and (iii) $v_t(\boldsymbol{p}^{\mathrm{A}})$ and $v_t(\boldsymbol{p}^{\mathrm{B}})$ are conditionally independent given $\mathcal{F}_\tau^m \oplus \mathcal{F}_{\tau-1}$ for any $\tau \leq t$. The terminal case ($t = T$) is trivial: p_t^{A} and p_t^{B} are both known constants and (i) and (ii) follow from the definition of v_T and $\boldsymbol{\theta}_{r,\mathrm{T}}$ and (ii) follows from the assumption of the proposition. Next, we assume that (i) –(iii) hold for time t and show they also hold for time t-1. From equation (7) and part (i) of the induction hypothesis, we have

$$v_{t-1}(\boldsymbol{p}^{\mathrm{A}} + \boldsymbol{p}^{\mathrm{B}}) = p_{t-1}^{\mathrm{A}} + p_{t-1}^{\mathrm{B}} + \frac{1}{(1+r_f)} \mathrm{E}^*[\mathrm{ECE}_t[\, v_t(\boldsymbol{p}^{\mathrm{A}}) + v_t(\boldsymbol{p}^{\mathrm{B}}) \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t-1}\,] \mid \mathcal{F}_{t-1}\,] \ .$$

Since, for the exponential utility, the certainty equivalent of the sum of two independent random variables ($v_t(\boldsymbol{p}^{\mathrm{A}})$ and $v_t(\boldsymbol{p}^{\mathrm{B}})$) is equal to the sum of the two certainty equivalents, we may rewrite this equation as:

$$v_{t-1}(\boldsymbol{p}^{\mathrm{A}} + \boldsymbol{p}^{\mathrm{A}}) = p_{t-1}^{\mathrm{A}} + p_{t-1}^{\mathrm{B}} + \frac{1}{(1+r_f)} \left(\mathrm{E}^*[\mathrm{ECE}_t[\, v_t(\boldsymbol{p}^{\mathrm{A}}) \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t-1}\,] \mid \mathcal{F}_{t-1}] + \mathrm{E}^*[\mathrm{ECE}_t[\, v_t(\boldsymbol{p}^{\mathrm{B}}) \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t-1}\,] \mid \mathcal{F}_{t-1}]\right)$$

or,

$$v_{t-1}(\boldsymbol{p}^{\mathrm{A}} + \boldsymbol{p}^{\mathrm{B}}) = v_{t-1}(\boldsymbol{p}^{\mathrm{A}}) + v_{t-1}(\boldsymbol{p}^{\mathrm{B}}).$$

This establishes part (i) of the induction hypothesis for time t-1 and part (ii) follows from the definition of $\boldsymbol{\theta}_{r,t}$ in equation (5). The induction hypothesis implies that $E^*[ECE_t[\, v_t(\boldsymbol{p}^A) \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t-1}\,] | \mathcal{F}_{t-1}]$ and $E^*[ECE_t[\, v_t(\boldsymbol{p}^B) \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t-1}\,] | \mathcal{F}_{t-1}]$ are conditionally independent given $\mathcal{F}_T^m$. Since p_{t-1}^A and p_{t-1}^B are similarly independent by the assumption of the proposition, $v_{t-1}(\boldsymbol{p}^A)$ and $v_{t-1}(\boldsymbol{p}^B)$ must also be independent, thereby establishing part (iii) for time t-1.///

Proposition 4: This result is immediate given the definitions and the result of Proposition 2.///

Lemma 2: (a) Substituting $\boldsymbol{\theta}_g = \boldsymbol{\theta}_s - \boldsymbol{\theta}_r + \boldsymbol{\theta}_b$ into the definition of $d_t(\boldsymbol{\theta}_g)$ and using $w_t = p_t - d_t(\boldsymbol{\theta}_r)$, we have

$$p_t + d_t(\boldsymbol{\theta}_g) = p_t + d_t(\boldsymbol{\theta}_s) - d_t(\boldsymbol{\theta}_r) + d_t(\boldsymbol{\theta}_b) = d_t(\boldsymbol{\theta}_s) + w_t + d_t(\boldsymbol{\theta}_b) \quad . \tag{A1}$$

The result for $t = 0$ follows from noting $d_0(\boldsymbol{\theta}_b) = 0$ and $w_0 = p_0 + \boldsymbol{\theta}_{r,0}\boldsymbol{s}_0 = v_0$. To prove the result for $t > 0$, we establish the following sequence of identities:

$$\begin{aligned}
d_t(\boldsymbol{\theta}_b) &= (\theta_{b,t-1,0} - \theta_{b,t,0})\,(1+r_f)^t \\
&= \left(\frac{R_t}{(1+r_f)^t}\left(\sum\nolimits_{\tau=1}^{t-1}\frac{w_\tau}{R_\tau}\right) - \frac{R_{t+1}}{(1+r_f)^{t+1}}\left(\sum\nolimits_{\tau=1}^{t}\frac{w_\tau}{R_\tau}\right)\right)(1+r_f)^t \\
&= \left(\left(\frac{R_t}{(1+r_f)^t} - \frac{R_{t+1}}{(1+r_f)^{t+1}}\right)\left(\sum\nolimits_{\tau=1}^{t}\frac{w_\tau}{R_\tau}\right) - \frac{R_t}{(1+r_f)^t}\frac{w_t}{R_t}\right)(1+r_f)^t \\
&= \rho_t \sum\nolimits_{\tau=1}^{t}\frac{w_\tau}{R_\tau} - w_t.
\end{aligned}$$

The first two equalities follow from the definition of the rebalancing portfolio. The third equality rearranges the windfall terms, factoring out $\sum_{\tau=1}^{t} w_\tau/R_\tau$ from both terms in the second line. The next equality is the result of canceling common risk tolerance terms in the third line. The next equality follows from multiplying through by the outer $(1+r_f)^t$, canceling common factors, and noting that $R_t - R_{t+1}/(1+r_f) = \rho_t$. Substituting this final expression back into (A1) yields the equation of the Lemma.

*** rewrite proof of (b) and sep theorem.

(b) The first equality of the proposition, $CE_t[\, \rho_\tau w_t/R_t \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t-1}] = ECE_t[w_t \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t-1}]$ follows from the definitions of the consumption certainty equivalent CE_t[-] and the effective certainty equivalent ECE_t[-]. Noting that $w_t = v_t - \boldsymbol{\theta}_{r,t-1}\boldsymbol{s}_t$ (as shown in the text in the paragraph before the Lemma), we can rewrite this effective certainty equivalent as

$$ECE_t[v_t - \boldsymbol{\theta}_{r,t-1}\boldsymbol{s}_t \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t-1}] \quad .$$

Noting that $\boldsymbol{\theta}_{r,t-1}\boldsymbol{s}_t$ is constant given $\mathcal{F}_t^m \oplus \mathcal{F}_{t-1}$ and exploiting the "Δ-property" of the exponential utility function (given an exponential utility function, for any constant Δ and gamble $\tilde{c}_t$, $ECE_t[\tilde{c}_t + \Delta] = ECE_t[\tilde{c}_t] + \Delta$), we can rewrite this as

$$ECE_t[v_t \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t-1}] - \boldsymbol{\theta}_{r,t-1}\boldsymbol{s}_t \,,$$

which is equal to zero by definition of the replicating portfolio given in equation (3).

(c) For $t > 0$, we have

$$\begin{aligned} \mathrm{CE}_t[p_t + d_t(\boldsymbol{\theta}_g)] &= \mathrm{CE}_t\left[d_t(\boldsymbol{\theta}_s) + \rho_t \sum_{\tau=1}^{t} \frac{w_\tau}{R_\tau} \right] \\ &= \mathrm{CE}_t\left[d_t(\boldsymbol{\theta}_s) + \rho_t \sum_{\tau=1}^{t-1} \frac{w_\tau}{R_\tau} + \mathrm{CE}_t\left[\rho_t \frac{w_t}{R_t} \mid \mathcal{F}_t^m \oplus \mathcal{F}_{t-1} \right] \right] \\ &= \mathrm{CE}_t\left[d_t(\boldsymbol{\theta}_s) + \rho_t \sum_{\tau=1}^{t-1} \frac{w_\tau}{R_\tau} \right] \end{aligned} \tag{A3}$$

The first equality follows from part (a) of this lemma. The second equality is derived by computing certainty equivalents iteratively (conditioning on the time-t market state and the time-(t-1) market state) and then applying the "Δ-property" of the exponential utility function (given an exponential utility function, for any constant Δ and gamble $\tilde{c}_t$, $\mathrm{CE}_t[\tilde{c}_t + \Delta] = \mathrm{CE}_t[\tilde{c}_t] + \Delta$) and noting that, for $\tau < t$, the windfalls w_τ, having been determined before period t are all known constants given $\mathcal{F}_{t-1}$; similarly $\boldsymbol{\theta}_s$ and $d_t(\boldsymbol{\theta}_s)$ are known given $\mathcal{F}_t^m$ and, using the "Δ-property", can be pulled outside the CE_t[-]. The next equality follows from part (b) of this lemma. At this point, we have eliminated the $\tau = t$ term from the summation in (A3). Repeating this process and taking expectations over the earlier private states (all conditioned on the time-t market state) we can eliminate the other terms to obtain the result of part (c). ///

Separation Theorem: We first prove the separation theorem in the special case where there is no flexibility in managing the project and then generalize to the case where there is a choice of production plans.

Let $\boldsymbol{p}$ be a fixed project with value v_0 and replicating and rebalancing trading strategies $\boldsymbol{\theta}_r$ and $\boldsymbol{\theta}_b$. Let $\boldsymbol{\theta}_s^*$ be an optimal solution to the portfolio-consumption problem. To establish the separation theorem for a fixed project, we need to show that $\boldsymbol{\theta}_g = \boldsymbol{\theta}_s^* - \boldsymbol{\theta}_r + \boldsymbol{\theta}_b$ is a solution to:

$$\max_{\boldsymbol{\theta}_g} \mathrm{E}\big[U(\boldsymbol{e} + \boldsymbol{p} + \boldsymbol{d}(\boldsymbol{\theta}_g))\big] \ .$$

Because U is assumed to be strictly concave, the first-order conditions for optimality are necessary and sufficient for a trading strategy $\boldsymbol{\theta}_g$ to be optimal. These first-order conditions may be written as

$$\frac{\partial}{\partial \boldsymbol{\theta}_{g,t}(\omega)} \mathrm{E}\big[U(\boldsymbol{e} + \boldsymbol{p} + \boldsymbol{d}(\boldsymbol{\theta}_g)) \mid \mathcal{F}_t\big] = \mathbf{0} \ ,$$

for all t and ω. Using the additive form of the investor's utility function, for $t > 0$, the first-order conditions are equivalent to

$$k_t \, \boldsymbol{s}_t \, u_t'(p_t + d_t(\boldsymbol{\theta}_g)) = k_{t+1} \, \mathrm{E}[\, \boldsymbol{s}_{t+1} \, u_{t+1}'(p_{t+1} + d_{t+1}(\boldsymbol{\theta}_g)) \mid \mathcal{F}_t] \tag{A4}$$

where u_t' denotes the derivative of the utility function u_t for period-t consumption. The $t = 0$ result is similar with the left side including $e_0 + p_0$ in place of p_0. The left side of (A4) is an n+1 vector whose entries represent the marginal utility of purchasing one additional share of each security at time t. The right side of (A4) is the corresponding vector with entries representing the expected marginal utilities generated by each additional share at time t+1. An optimal trading strategy equates these two vectors of marginal utilities.

In the remainder of the proof, we focus on the case where $t > 0$; the proof for $t = 0$ is similar. Taking $\boldsymbol{\theta}_g = \boldsymbol{\theta}_s^* - \boldsymbol{\theta}_r + \boldsymbol{\theta}_b$, rewriting (A4) using part (a) of Lemma 2, and dropping common constant factors, we see that (A4) holds if and only if:

$$k_t\, \boldsymbol{s}_t\, u_t'(d_t(\boldsymbol{\theta}_s^*)) \;=\; k_{t+1}\, \mathrm{E}\left[\, \boldsymbol{s}_{t+1}\, u_{t+1}'(d_{t+1}(\boldsymbol{\theta}_s^*))\, exp\left(-\frac{p_{t+1} - d_{t+1}(\boldsymbol{\theta}_r)}{R_{t+1}}\right) \mid \mathcal{F}_t\right] \tag{A5}$$

Noting that $\boldsymbol{s}_{t+1}$ and $d_{t+1}(\boldsymbol{\theta}_s{}^*)$ are both constants given the time-(t+1) market state and taking expectations iteratively, we can rewrite the right side of (A5) as:

$$\mathrm{E}\left[\, \boldsymbol{s}_{t+1}\, u_{t+1}'(d_{t+1}(\boldsymbol{\theta}_s^*))\, \mathrm{E}\left[exp\left(-\frac{p_{t+1} - d_{t+1}(\boldsymbol{\theta}_r)}{R_{t+1}}\right) \mid \mathcal{F}_{t+1}^m \oplus \mathcal{F}_t\right] \mid \mathcal{F}_t\right]$$

Applying part (b) of the lemma (in particular equation A2), this becomes

$$\mathrm{E}[\, \boldsymbol{s}_{t+1}\, u_{t+1}'(d_{t+1}(\boldsymbol{\theta}_s^*)) \mid \mathcal{F}_t] \;.$$

Noting that this expression is independent of the private states and substituting back into (A5), we find that (A4) is satisfied if

$$k_t\, \boldsymbol{s}_t\, u_t'(d_t(\boldsymbol{\theta}_s^*)) \;=\; k_{t+1}\, \mathrm{E}[\, \boldsymbol{s}_{t+1}\, u_{t+1}'(d_{t+1}(\boldsymbol{\theta}_s^*)) \mid \mathcal{F}_t^m\,] \;\;.$$

These are precisely the first-order necessary and sufficient conditions for the portfolio-consumption problem (they are analogous to the conditions of equation A4). Thus if $\boldsymbol{\theta}_s^*$ is optimal for the portfolio-consumption problem, then $\boldsymbol{\theta}_g = \boldsymbol{\theta}_s^* - \boldsymbol{\theta}_r + \boldsymbol{\theta}_b$ is optimal for the grand problem.

Having established the separation theorem for a fixed project, we now show that the production plan $\boldsymbol{\pi}^*$ that maximizes the project value v_0 is optimal for the grand problem (1). To do this note that, for any fixed project, Lemma 2 (parts a and c) imply that the certainty equivalents for each period's consumption are identical in the portfolio-consumption and grand problems. Because the utility function U is additive, this then implies that, for any fixed project, the maximal overall expected utilities are the same as well. Since the utility function is increasing in first period consumption, given flexibility in choosing a production plan, any increase in the project value v_0 leads to an increase in the expected utility in the portfolio-consumption problem and, hence, in the grand problem as well. Thus the production plan $\boldsymbol{\pi}^*$ that maximizes the present value v_0 is optimal for the grand problem. ///

Proposition 5: Our proof focuses on the upper bound, a similar proof holds for the lower bound. Let $\mathcal{U}_t(e_t) \equiv U_t(e_t, \mathbf{0})$ denote the investor's "true" utility for wealth assuming he forgoes all future project income, and let $\bar{\mathcal{U}}_t(e_t) = -\exp(-e_t/\bar{R}_t)$ be the corresponding utility function given by assuming constant risk tolerances $\bar{\rho}_\tau$ for all $\tau \geq t$; $\bar{R}_t$ is the upper bound effective risk tolerance defined as in equation (5). We proceed in two steps.

Part 1: We first show that $\mathcal{U}_t$ is "more risk averse" than $\bar{\mathcal{U}}_t$ in that for any $\mathcal{F}$Ăadapted random variable $\tilde{x}_t$ and time-t wealth e_t such that the possible values of $e_t + \tilde{x}_t$ are within the range of possible period-t wealth levels (i.e., min $\{\boldsymbol{\theta}_{g,t}\, \boldsymbol{s}_{t+1} + v_{t+1}\} \leq \tilde{x}_t \leq$ max $\{\boldsymbol{\theta}_{g,t}\, \boldsymbol{s}_{t+1} + v_{t+1}\}$)), the certainty equivalent x_t^c given by the true utility function is less than the corresponding certainty equivalent $\bar{x}_t^c$ given by the upper bound utility function. The certainty equivalents x_t^c and $\bar{x}_t^c$ are defined as $\mathcal{F}_t$-measurable random variables such that $\mathcal{U}_t(e_t + \bar{x}_t^c) = \mathrm{E}[\mathcal{U}_t(e_t + \tilde{x}_t) \mid \mathcal{F}_t]$ and $\bar{\mathcal{U}}_t(e_t + \bar{x}_t^c) = \mathrm{E}[\bar{\mathcal{U}}_t(e_t + \tilde{x}_t) \mid \mathcal{F}_t]$. We prove $x_t^c \leq \bar{x}_t^c$ by induction. The terminal case is straightforward: $\mathcal{U}_T(e_T) = k_T u_T(e_T)$ and $\bar{\mathcal{U}}_t(e_t) = -\exp(-e_t/\bar{\rho}_T)$, so $x_T^c \leq \bar{x}_T^c$ follows from the

definition of $\bar{\rho}_T$ as an upper bound on the period risk tolerance (using, e.g., Theorem 1 of Pratt 1964). We now assume that $x_\tau^c \le \bar{x}_\tau^c$ holds for all $\tau > t$ and show that it holds for period t as well.

Let $\boldsymbol{\theta}_t^b$ denote the optimal portfolio given x_t^c as a lump-sum in period t, i.e., the solution to

$$\mathcal{U}_t(e_t + x_t^c) = \max_{\boldsymbol{\theta}_t} \left\{ k_t u_t(e_t + x_t^c - \boldsymbol{\theta}_t \boldsymbol{s}_t) + \mathrm{E}[\mathcal{U}_{t+1}(\boldsymbol{\theta}_t \boldsymbol{s}_{t+1}) \,|\mathcal{F}_t] \right\} \tag{A6}$$

Note that though the trading strategy $\boldsymbol{\theta}_t^b$ need not be unique, because of the strict convexity of the utility functions u_t and $\mathcal{U}_{t+1}$, the pattern of consumption generated by the trading strategy will be unique. Let $\boldsymbol{\theta}_t^*(x_t)$ be a trading strategy such that $\boldsymbol{\theta}_t^b + \boldsymbol{\theta}_t^*(\tilde{x}_t)$ is an optimal trading strategy given outcome x_t of the gamble (i.e., with x_t in place of x_t^c in A7); $\boldsymbol{\theta}_t^*(\tilde{x}_t)$ can be interpreted as an adjustment in the trading strategy in response to the outcome of the gamble. We then have

$$\begin{aligned}
\mathcal{U}_t(e_t + x_t^c) &= \mathrm{E}[\, \mathcal{U}_t(e_t + \tilde{x}_t) \,|\, \mathcal{F}_t] \\
&= \mathrm{E}\big[k_t u_t(e_t + \tilde{x}_t - (\boldsymbol{\theta}_t^b + \boldsymbol{\theta}_t^*(\tilde{x}_t))\boldsymbol{s}_t) \,|\, \mathcal{F}_t\big] + \mathrm{E}\big[\, \mathrm{E}[\mathcal{U}_{t+1}((\boldsymbol{\theta}_t^b + \boldsymbol{\theta}_t^*(\tilde{x}_t))\boldsymbol{s}_{t+1}) \,|\, \mathcal{F}_{t+1}^m \oplus \mathcal{F}_t] \,|\, \mathcal{F}_t\big] \\
&= k_t u_t(e_t + \hat{x}_t - \boldsymbol{\theta}_t^b \boldsymbol{s}_t) + \mathrm{E}[\mathcal{U}_{t+1}(\boldsymbol{\theta}_t^b \boldsymbol{s}_{t+1} + \hat{x}_{t+1}) \,|\, \mathcal{F}_t] \;.
\end{aligned}$$

The first equality follows from the definition of v_t as the effective certainty equivalent of $\tilde{x}_t$ and the second from the definitions of $\mathcal{U}_t$, $\boldsymbol{\theta}_t^b$, and $\boldsymbol{\theta}_t^*(\tilde{x}_t)$, and taking expectations iteratively in the second term. In the next line, we replace the expressions depending on the gamble $\tilde{x}_t$ by their certainty equivalents: $\hat{x}_t$ is the $\mathcal{F}_t$-measurable random variable such that $u_t(e_t + \hat{x}_t - \boldsymbol{\theta}_t^b \boldsymbol{s}_t) = \mathrm{E}\big[u_t(e_t + x_t - (\boldsymbol{\theta}_t^b + \boldsymbol{\theta}_t^*(\tilde{x}_t))\boldsymbol{s}_t) \,|\, \mathcal{F}_t\big]$ and $\hat{x}_{t+1}$ is the $(\mathcal{F}_{t+1}^m \oplus \mathcal{F}_t)$-measurable random variable such that $\mathcal{U}_{t+1}(\boldsymbol{\theta}_t^b \boldsymbol{s}_{t+1} + \hat{x}_{t+1}) = \mathrm{E}[\mathcal{U}_{t+1}((\boldsymbol{\theta}_t^b + \boldsymbol{\theta}_t^*(\tilde{x}_t))\boldsymbol{s}_{t+1}) \,|\mathcal{F}_{t+1}^m \oplus \mathcal{F}_t]$. Because $\hat{x}_{t+1}$ is $(\mathcal{F}_{t+1}^m \oplus \mathcal{F}_t)$-measurable, the consumption stream in the third line can be achieved by trading securities without conditioning on the outcome of the gamble and hence is feasible for (A6). Since this consumption stream achieves the maximum utility in equation (A6) and the optimal consumption pattern in (A6) is unique, it follows that $\hat{x}_t = x_t^c$ and $\hat{x}_{t+1} = 0$.

Now, let $\bar{\boldsymbol{\theta}}_t^b$ denote the optimal portfolio for the upper bound utility given x_t^c as a lump-sum in period t, i.e., the solution to

$$\bar{\mathcal{U}}_t(e_t + x_t^c) = \max_{\boldsymbol{\theta}_t} \left\{ k_t \bar{u}_t(e_t + x_t^c - \boldsymbol{\theta}_t \boldsymbol{s}_t) + \mathrm{E}[\bar{\mathcal{U}}_{t+1}(\boldsymbol{\theta}_t \boldsymbol{s}_{t+1}) \,|\, \mathcal{F}_t] \right\} .$$

Then we have

$$\begin{aligned}
\bar{\mathcal{U}}_t(e_t + \bar{x}_t^c) &= \mathrm{E}[\, \bar{\mathcal{U}}_t(e_t + \tilde{x}_t) \,|\, \mathcal{F}_t] \\
&\ge \mathrm{E}\big[\, k_t\, \bar{u}_t(e_t + \tilde{x}_t - (\bar{\boldsymbol{\theta}}_t^b + \boldsymbol{\theta}_t^*(x))\boldsymbol{s}_t) \,|\, \mathcal{F}_t\big] + \mathrm{E}\big[\, \mathrm{E}[\bar{\mathcal{U}}_{t+1}((\bar{\boldsymbol{\theta}}_t^b + \boldsymbol{\theta}_t^*(x))\boldsymbol{s}_{t+1}) \,|\, \mathcal{F}_{t+1}^m \oplus \mathcal{F}_t] \,|\, \mathcal{F}_t\big] \\
&\ge k_t \bar{u}_t(e_t + \hat{x}_t - \bar{\boldsymbol{\theta}}_t^b \boldsymbol{s}_t) + \mathrm{E}[\bar{\mathcal{U}}_{t+1}(\bar{\boldsymbol{\theta}}_t^b \boldsymbol{s}_{t+1} + \hat{x}_{t+1}) \,|\, \mathcal{F}_t] \\
&= k_t \bar{u}_t(e_t + x_t^c - \bar{\boldsymbol{\theta}}_t^b \boldsymbol{s}_t) + \mathrm{E}[\bar{\mathcal{U}}_{t+1}(\bar{\boldsymbol{\theta}}_t^b \boldsymbol{s}_{t+1}) \,|\, \mathcal{F}_t] \\
&= \bar{\mathcal{U}}_t(e_t + x_t^c) \;.
\end{aligned}$$

The first equality follows from the definition of $\bar{x}_t^c$ as a certainty equivalent. The next inequality follows from the definition of $\bar{\mathcal{U}}_t$ and noting that $\bar{\boldsymbol{\theta}}_t^b + \boldsymbol{\theta}_t^*(\tilde{x}_t)$ is a feasible but non-optimal trading strategy given the outcome of the gamble; the adjustment $\boldsymbol{\theta}_t^*(\tilde{x}_t)$ is optimal for the true preferences but not the upper

bound preferences. The third line follows from rearranging the order of expectations as before. The next inequality (line 4) follows from $\bar{u}_t$ and $\bar{\mathcal{U}}_{t+1}$ being less risk averse than u_t and $\mathcal{U}_{t+1}$ (by the assumption of the proposition and the induction hypothesis), and hence yield certainty equivalents in each period greater than those given by the true utility ($\hat{x}_t$ and $\hat{x}_{t+1}$). The consumption in period t ranges over the set of possible consumption levels identified in the proposition provided the outcomes of the gamble $\tilde{x}_t$ ranges over the set of possible values v_t. The next equality follows from $\hat{x}_t = x_t^c$ and $\hat{x}_{t+1} = 0$ (as established in the previous paragraph). The final line follows from the definition of $\bar{\boldsymbol{\theta}}_t^*$. Thus we have established $\bar{\mathcal{U}}_t(e_t + \bar{x}_t^c) \geq \bar{\mathcal{U}}_t(e_t + x_t^c)$, which implies that the certainty equivalents for the gamble $\tilde{x}_t$ satisfy $x_t^c \leq \bar{x}_t^c$.

Part II. We now use the result of Part I to show that the true and upper bound present certainty equivalent values satisfy $v_t \leq \bar{v}_t$. Again the proof is by induction with the terminal case ($t = T$) being trivial as there is no uncertainty and $v_T = \bar{v}_T$. Now assuming that $v_\tau \leq \bar{v}_t$ holds for all $\tau > t$, we show that they hold for time t as well. We establish this result as follows:

$$\mathcal{U}_t(e_t + v_t) = U_t(e_t, \boldsymbol{p}) \tag{A7.1}$$

$$= k_t u_t(e_t + p_t - \boldsymbol{\theta}^*_{g,t}\, \boldsymbol{s}_t) + \mathrm{E}[U_{t+1}(\boldsymbol{\theta}^*_{g,t}\, \boldsymbol{s}_{t+1}, \boldsymbol{p}) \mid \mathcal{F}_t] \tag{A7.2}$$

$$= k_t u_t(e_t + p_t - \boldsymbol{\theta}^*_{g,t}\, \boldsymbol{s}_t) + \mathrm{E}[\mathcal{U}_{t+1}(\boldsymbol{\theta}^*_{g,t}\, \boldsymbol{s}_{t+1} + v_{t+1}) \mid \mathcal{F}_t] \tag{A7.3}$$

$$= k_t u_t(e_t + p_t - \boldsymbol{\theta}^*_{g,t}\, \boldsymbol{s}_t) + \mathrm{E}[\ \mathrm{E}[\mathcal{U}_{t+1}(\boldsymbol{\theta}^*_{g,t}\, \boldsymbol{s}_{t+1} + v_{t+1}) \mid \mathcal{F}^m_{t+1} \oplus \mathcal{F}_t] \mid \mathcal{F}_t] \tag{A7.4}$$

$$\leq k_t u_t(e_t + p_t - \boldsymbol{\theta}^*_{g,t}\, \boldsymbol{s}_t) + \mathrm{E}[\mathcal{U}_{t+1}(\boldsymbol{\theta}^*_{g,t}\, \boldsymbol{s}_{t+1} + \mathrm{ECE}_{t+1}[\ v_{t+1} \mid \mathcal{F}^m_{t+1} \oplus \mathcal{F}_t]) \mid \mathcal{F}_t] \tag{A7.5}$$

$$\leq k_t u_t(e_t + p_t - \boldsymbol{\theta}^*_{g,t}\, \boldsymbol{s}_t) + \mathrm{E}[\mathcal{U}_{t+1}(\boldsymbol{\theta}^*_{g,t}\, \boldsymbol{s}_{t+1} + \mathrm{ECE}_{t+1}[\ \bar{v}_{t+1} \mid \mathcal{F}^m_{t+1} \oplus \mathcal{F}_t]) \mid \mathcal{F}_t] \tag{A7.6}$$

$$= k_t u_t(e_t + \bar{v}_t - (\boldsymbol{\theta}^*_{g,t} + \boldsymbol{\theta}_{r,t})\boldsymbol{s}_t) + \mathrm{E}[\mathcal{U}_{t+1}((\boldsymbol{\theta}^*_{g,t} + \boldsymbol{\theta}_{r,t})\boldsymbol{s}_{t+1}) \mid \mathcal{F}_t] \tag{A7.7}$$

$$\leq \max_{\boldsymbol{\theta}_t} \left\{\ k_t u_t(e_t + \bar{v}_t - \boldsymbol{\theta}_t \boldsymbol{s}_t) + \mathrm{E}[\mathcal{U}_{t+1}(\boldsymbol{\theta}_t \boldsymbol{s}_{t+1}) \mid \mathcal{F}_t]\ \right\} \tag{A7.8}$$

$$= \mathcal{U}_t(e_t + \bar{v}_t) \tag{A7.9}$$

The first line is the definition of the period-t value v_t, the second follows from the definition of $\boldsymbol{\theta}^*_{g,t}$, and the third from the definition of the period-(t+1) value v_t. The fourth line follows by taking expectations iteratively. In the fifth line, we let $\mathrm{ECE}_{t+1}[\ - \mid \mathcal{F}^m_{t+1} \oplus \mathcal{F}_t]$ denote the effective certainty equivalent (defined in equation 4) given by assuming constant risk tolerances $\bar{\rho}_\tau$ for all $\tau > t$. The inequality of the fifth line then follows from the result of part I and the inequality of the sixth line follows from the induction hypothesis. Taking $\boldsymbol{\theta}_{r,t}$ to be a certainty-equivalent replicating portfolio defined (as in equation 3) as solving $\boldsymbol{\theta}_{r,t}\, \boldsymbol{s}_{t+1} = \mathrm{ECE}_{t+1}[\ \bar{v}_{t+1} \mid \mathcal{F}^m_{t+1} \oplus \mathcal{F}_t]$, the seventh line then follows using this along with the definition of $\bar{v}_t$ as $p_t + \boldsymbol{\theta}_{r,t}\, \boldsymbol{s}_t$. The next line follows since $(\boldsymbol{\theta}^*_{g,t} + \boldsymbol{\theta}_{r,t})$ is a feasible, but generally non-optimal, portfolio for the optimization problem of line 8. The final equality follows from the definition of $\mathcal{U}_t$. We have thus established $\mathcal{U}_t(w_t + v_t) \leq \mathcal{U}_t(w_t + \bar{v}_t)$ which implies $v_t \leq \bar{v}_t$ as desired. ///

Proposition 6: In a Pareto optimal sharing arrangement, the distribution of wealth in each period (after that period's uncertainties are resolved) would maximize a weighted sum of the individual's expected utilities for wealth in that period. It is easiest to focus on a particular state of information $S_{t\text{-}1}$ and consider the arrangement to share a gamble x_t^0 that is resolved in the next period and pays $x_t^0(S_t)$ in each

state $S_t \in \mathcal{P}_t$ such that $S_t \subseteq S_{t-1}$. A Pareto optimal sharing arrangement for a gamble $x_t^0(S_t)$ is a set of functions $(x_t^1, x_t^2, \ldots, x_t^I)$ with $x_t^i(S_t)$ specifying an allocation that of wealth to investor i in period t and state S_t, such that, for some set of weights $(\lambda^1, \lambda^2, \ldots, \lambda^I)$, $\lambda^i > 0$, solves

$$\max_{(x_t^1, x_t^2, \ldots, x_t^I)} \sum_{i=1}^{I} \sum_{S_t \subseteq S_{t-1}} \lambda^i P^i(S_t \mid S_{t-1}) \mathcal{U}_t^i(x_t^i(S_t)) \tag{A9}$$

$$\text{subject to } \sum_{i=1}^{I} x_t^i(S_t) = x_t^0(S_t) \quad \text{for all } S_t \subseteq S_{t-1} \ ,$$

where $\mathcal{U}_t^i$ denotes investor i's utility for wealth in period t. A Pareto efficient group would then evaluate alternative gambles x_t^0 on the basis of the maximal value of the objective function in (A9) for a given set of weights.

From the separation theorem, we know that $\mathcal{U}_t^i(x_t^i)$ can be represented as $-\exp(-x_t^i/R_t^i)$ where R_t^i is investor i's period-t effective risk tolerance. The first-order conditions for (A9) require the existence of a set of Lagrange multipliers $\mu_t(S_t)$ such that, for all i and states $S_t \subseteq S_{t-1}$, the optimal sharing arrangement x_t^i * satisfies:

$$\lambda^i P^i(S_t \mid S_{t-1}) \exp\left(-\frac{x_t^{i*}(S_t)}{R_t^i}\right) = \mu_t(S_t) R_t^i \ . \tag{A10}$$

Solving this for $x_t^i{*}(S_t)$, we find

$$x_t^i{*}(S_t) = -R_t^i \left\{ \ln(\mu_t(S_t)) - \ln\left(\frac{\lambda^i}{R_t^i}\right) - \ln(P^i(S_t \mid S_{t-1})) \right\} .$$

Summing over i and using the constraint from (A9), this implies

$$x_t^0(S_t) = -R_t^0 \left\{ \ln(\mu_t(S_t)) - \ln\left(\prod_{i=1}^{I} \left(\frac{\lambda^i}{R_t^i}\right)^{R_t^i/R_t^0}\right) - \ln\left(\prod_{i=1}^{I} P^i(S_t \mid S_{t-1})^{R_t^i/R_t^0}\right) \right\}$$

or, rearranging,

$$\lambda^0 P_t^0(S_t \mid S_{t-1}) \exp\left(-\frac{x_t^0(S_t)}{R_t^0}\right) = \mu_t(S_t) R_t^0 \tag{A11}$$

where $P_t^0(S_t \mid S_{t-1}) = \prod_{i=1}^{I} P_t^i(S_t \mid S_{t-1})^{R_t^i/R_t^0}$ and $\lambda^0 = R_t^0 \prod_{i=1}^{I} (\lambda^i/R_t^i)^{R_t^i/R_t^0}$. Using this with (A10), we can write the objective function in (A9) as

$$\sum_{S_t \subseteq S_{t-1}} -\lambda^0 P_t^0(S_t \mid S_{t-1}) \exp\left(-\frac{x_t^0(S_t)}{R_t^0}\right) \ , \tag{A12}$$

where λ^0 enters only as an irrelevant constant. Therefore, the group of investors evaluates gambles in each period as if it were a single investor with probabilities $P_t^0(S_t \mid S_{t-1})$ and an exponential utility with

effective risk tolerance R_t^0, and would evaluate production plans using the valuation procedure of Section 3.

To complete the proof of part (a) of the proposition, we need to show that the aggregate probabilities for private events used in the valuation procedure ($P_t^0(S_t \mid S_t^m, S_{t-1})$) are given by aggregating the individual probabilities for private events. To see this, note that, since $P_t^i(S_t \mid S_{t-1}) = P_t^i(S_t \mid S_t^m, S_{t-1})\, P_t^i(S_t^m \mid S_{t-1})$,

$$P_t^0(S_t \mid S_{t-1}) = \prod_{i=1}^{I} P_t^i(S_t \mid S_t^m, S_{t-1})^{R_t^i/R_t^0} \prod_{i=1}^{I} P_t^i(S_t^m \mid S_{t-1})^{R_t^i/R_t^0} = P_t^0(S_t \mid S_t^m, S_{t-1})\, P_t^0(S_t^m \mid S_{t-1})$$

where $P_t^0(S_t \mid S_t^m, S_{t-1})$ and $P_t^0(S_t^m \mid S_{t-1})$ are aggregate probabilities for private and market events respectively, the latter being irrelevant in the valuation procedure of section 3.

To establish part (b) of the proposition, combining (A10) and (A11) we find

$$\frac{1}{R_t^i}\, \lambda^i\, P^i(S_t \mid S_{t-1}) \exp\left(-\frac{x_t^i{*}(S_t)}{R_t^i}\right) = \frac{1}{R_t^0}\, \lambda^0\, P_t^0(S_t \mid S_{t-1}) \exp\left(-\frac{x_t^0(S_t)}{R_t^0}\right) \quad .$$

Solving for $x_t^i{*}(S_t)$ yields

$$x_t^i{*}(S_t) = R_t^i \ln\left(\frac{\lambda^i R_t^0}{\lambda^0 R_t^i}\right) + R_t^i \ln\left(\frac{P_t^i(S_t \mid S_{t-1})}{P_t^0(S_t \mid S_{t-1})}\right) + \frac{R_t^i}{R_t^0}\, x_t^0(S_t) \qquad .$$

The first term here totals zero when summed over investors and can be interpreted as deterministic "side payment"; the amount of the payment depends on the weights (λ^i) associated with investor *i*, but is independent of the gamble (x_t^0) or the beliefs of the investors. The second term also sums to zero and can be interpreted as a "side bet" that depends on the investors' beliefs, but not the project or weights; intuitively, investor *i* receives more in those scenarios where his probability is higher than the aggregate probability. The final term represents investor *i*'s share of the gamble x_t^0 and we see that each investor shares in proportion to their effective risk tolerance. In the context of the valuation procedure, the changes in wealth in each period are specified by the windfalls w_t, and, hence, each investor would share these windfalls in proportion to there effective risk tolerances, as specified in part (b) of the proposition.

The unanimity claim of part (c) follows from the fact that the evaluation of projects are independent of the weights λ^i associated the individuals. Thus, if we assigned vanishingly small weights to all individuals except individual *i* (as individual *i* would want to do), we would reach the same conclusions.///

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