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Online Appendices for

Information Relaxations and Duality in Stochastic Dynamic Programs

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A. Proofs and Supplemental Discussion

A.1. Proof of Theorem 2.1

Proof. By weak duality, the right side of (7) is greater than or equal to the left side. To establish strong duality, we need to show that if the left side is bounded, there exists a z^* that obtains equality. Let $z^*(a) = r(a) - v^*$ where v^* is the optimal value of the primal DP (1). To see that this z is dual feasible, note that for $\alpha_F \in \mathcal{A}_{\mathbb{F}}$, $\mathbb{E}[z^*(\alpha_F)] = \mathbb{E}[r(\alpha_F)] - v^*$. Because $\mathbb{E}[r(\alpha_F)] \leq v^*$ (by definition of v^* as the supremum over policies in $\mathcal{A}_{\mathbb{F}}$), we know that $\mathbb{E}[z^*(\alpha)] \leq 0$ for all $\alpha_F \in \mathcal{A}_{\mathbb{F}}$; thus z^* is dual feasible. With this penalty, the penalized objective $r(a) - z(a)$ is equal to v^* for all a and hence, for any policy α (including those in $\mathcal{A}_{\mathbb{G}}$), $\mathbb{E}[r(\alpha) - z^*(\alpha)] = v^*$. This implies that z^* achieves equality in (7).

If the primal problem is unbounded, by weak duality, the dual problem must also be unbounded. □

A.2. Proof of Theorem 2.2

Proof. We first consider sufficiency. Consider any $\alpha_F^* \in \mathcal{A}_{\mathbb{F}}$ and $z^* \in \mathcal{Z}_{\mathbb{F}}$ and suppose (8) holds and $\mathbb{E}[z^*(\alpha_F^*)] = 0$. Then we can rewrite the dual problem with this penalty as

$$\begin{aligned}
\sup_{\alpha_G \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}[r(\alpha) - z^*(\alpha)] &= \mathbb{E}[r(\alpha_F^*) - z^*(\alpha_F^*)] && \text{(using (8))} \\
&= \mathbb{E}[r(\alpha_F^*)] && \text{(since } \mathbb{E}[z^*(\alpha_F^*)] = 0\text{).}
\end{aligned}$$

Then, by weak duality, α_F^* and z^* must be optimal.

To show necessity, first note that for any $\alpha_F^* \in \mathcal{A}_{\mathbb{F}}$ and $z^* \in \mathcal{Z}_{\mathbb{F}}$, we have:

$$\begin{aligned}
\sup_{\alpha_G \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}[r(\alpha_G) - z^*(\alpha_G)] &\geq \sup_{\alpha_F \in \mathcal{A}_{\mathbb{F}}} \mathbb{E}[r(\alpha_F) - z^*(\alpha_F)] && \text{(because } \mathcal{A}_{\mathbb{F}} \subseteq \mathcal{A}_{\mathbb{G}}\text{)} \\
&\geq \mathbb{E}[r(\alpha_F^*) - z^*(\alpha_F^*)] && \text{(because } \alpha_F^* \in \mathcal{A}_{\mathbb{F}}\text{)} \\
&\geq \mathbb{E}[r(\alpha_F^*)] && \text{(because } z^* \in \mathcal{Z}_{\mathbb{F}}\text{).}
\end{aligned}$$

If $\alpha_F^* \in \mathcal{A}_{\mathbb{F}}$ and $z^* \in \mathcal{Z}_{\mathbb{F}}$ are primal and dual optimal (respectively), then by the strong duality theorem, the first and last terms above are equal, so the intervening inequalities must hold with equality and we have $\mathbb{E}[z^*(\alpha_F^*)] = 0$ and (8). □

A.3. Proof of Proposition 2.1

Proof. The first inequality in (9) follows from applying the weak duality result (Lemma 2.1) with the restricted policy space $\mathcal{S}$ in place of the full policy space $\mathcal{A}$. Note that, by definition, any dual feasible penalty z for the original problem with $\mathcal{A}$ satisfies $\mathbb{E}[z(\alpha_F)] \leq 0$ for all α_F in $\mathcal{A}_{\mathbb{F}}$. Since $\mathcal{S} \subseteq \mathcal{A}$, any such penalty will also be dual feasible with a restricted policy space, i.e, $\mathbb{E}[z(\alpha_F)] \leq 0$ for all α_F in $\mathcal{S}_{\mathbb{F}}$. This set of dual feasible penalties in the restricted policy space is larger than the set of dual feasible penalties in original space. Thus this first inequality holds on the larger set of penalties that are dual feasible with the restricted penalties.

The second inequality in (9) follows from the fact that $\mathcal{S}_{\mathbb{G}} \subseteq \mathcal{A}_{\mathbb{G}}$. □

A.4. Proof of Proposition 2.2

Before proving Proposition 2.2, we first establish a lemma that is used in the proof of this proposition as well as some later results.

Lemma A.1. *Let $z_t^{\mathbb{G}}(a) = \mathbb{E}[w_t(a)|\mathcal{G}_t]$. If $w_t(a)$ depends on the first t+1 actions in a and α_G is $\mathbb{G}$-adapted, then $z_t^{\mathbb{G}}(\alpha_G) = \mathbb{E}[w_t(\alpha_G)|\mathcal{G}_t]$, almost everywhere.*

Note that this result need not hold for policies that are not $\mathbb{G}$-adapted: In $z_t^{\mathbb{G}}(\alpha_G)$, the policy α_G selects actions a for a given ω and $z_t^{\mathbb{G}}(\alpha_G)$ takes on the corresponding value of $\mathbb{E}[w_t(a)|\mathcal{G}_t]$. That is, we calculate the "$\mathbb{G}$-average" in the conditional expectation first and then select averaged values. In $\mathbb{E}[w_t(\alpha_G)|\mathcal{G}_t]$ we select values $w_t(a)$ according to the policy α_G first and then calculate the averages in the conditional expectations. In these terms, the lemma says that if α_G is $\mathbb{G}$-adapted, $\mathbb{G}$-averaging then selecting is equivalent to selecting then $\mathbb{G}$-averaging.

Proof. Consider a $\mathbb{G}$-adapted policy α_G and pick an ω^0 in Ω and let $a^0 = \alpha_G(\omega^0)$. Define H^0 as the set of ω such that the first $t+1$ actions in a match those of a^0; note that ω^0 is in H^0. Because α_G is $\mathbb{G}$-adapted, we know $H^0 \in \mathcal{G}_t$. For any set $H \in \mathcal{G}_t$ such that $H \subseteq H^0$, we have

$$\int_H z_t^{\mathbb{G}}(\alpha_G) d\mathbb{P} = \int_H \mathbb{E}\left[w_t(a^0)|\mathcal{G}_t\right] d\mathbb{P} = \int_H w_t(a^0) d\mathbb{P}.$$

The first equality follows from the definition of $z_t^{\mathbb{G}}(\alpha_G)$ taking into account the fact $H \subseteq H^0$ and that, using α_G, all ω in H^0 select the same actions $(a_0, \ldots, a_t)$ as a^0. The second equality follows from the definition of conditional expectations (see, e.g., Billingsley 1986, p. 466)[1] using the fact that $H \in \mathcal{G}_t$. Similarly, for $H \in \mathcal{G}_t$ such that $H \subseteq H^0$, we have

$$\int_H \mathbb{E}\left[w_t(\alpha_G)|\mathcal{G}_t\right] d\mathbb{P} = \int_H w_t(\alpha_G) d\mathbb{P} = \int_H w_t(a^0) d\mathbb{P}.$$

Here we first use the definition of the conditional expectations and then use the fact that, under α_G, all ω in H^0 select the same actions $(a_0, \ldots, a_t)$ as a^0. Thus for $H \in \mathcal{G}_t$ such that $H \subseteq H^0$, we have

$$\int_H z_t^{\mathbb{G}}(\alpha_G) d\mathbb{P} = \int_H \mathbb{E}\left[w_t(\alpha_G)|\mathcal{G}_t\right] d\mathbb{P}. \tag{25}$$

We now show (25) is sufficient to ensure that $z_t^{\mathbb{G}}(\alpha_G) = \mathbb{E}[w_t(\alpha_G)|\mathcal{G}_t]$ for almost all $\omega \in H$. Note that $z_t^{\mathbb{G}}(\alpha_G)$ and $\mathbb{E}[w_t(\alpha_G)|\mathcal{G}_t]$ are both $\mathcal{G}_t$-measurable; thus $f = z_t^{\mathbb{G}}(\alpha_G) - \mathbb{E}[w_t(\alpha_G)|\mathcal{G}_t]$ is also $\mathcal{G}_t$-measurable. Let H^+ be the subset of H^0 where f is strictly positive; that is the subset of H^0 where $z_t^{\mathbb{G}}(\alpha_G) > \mathbb{E}[w_t(\alpha_G)|\mathcal{G}_t]$. Because f is $\mathcal{G}_t$-measurable, $H^+ \in \mathcal{G}_t$. Then from (25), we know that $\int_{H^+} f d\mathbb{P} = 0$, which (because $f > 0$ on H^+) implies that H^+ has measure 0 (see, e.g., Billingsley 1986 p. 466). We can similarly define a set H^- where $f < 0$ and conclude that H^- has measure 0. Thus, we can conclude that $f = 0$ or $z_t^{\mathbb{G}}(\alpha_G) = \mathbb{E}[w_t(\alpha_G)|\mathcal{G}_t]$ holds almost surely on H^0. Since we can construct such a set H^0 containing ω^0 for any ω^0, we can conclude that $z_t^{\mathbb{G}}(\alpha_G) = \mathbb{E}[w_t(\alpha_G)|\mathcal{G}_t]$ for almost all ω. □

We now turn to the proof of Proposition 2.2.

Proof. (i) We can write $z_t(a) = z_t^{\mathbb{G}}(a) - z_t^{\mathbb{F}}(a)$ where $z_t^{\mathbb{G}}(a) = \mathbb{E}[w_t(a)|\mathcal{G}_t]$ and $z_t^{\mathbb{F}}(a) = \mathbb{E}[w_t(a)|\mathcal{F}_t]$. Any α_F that is $\mathbb{F}$-adapted is also $\mathbb{G}$-adapted because $\mathbb{G}$ is a relaxation of $\mathbb{F}$. We can then write

$$\mathbb{E}\left[z_t^{\mathbb{G}}(\alpha_F)|\mathcal{F}_t\right] = \mathbb{E}\left[\mathbb{E}\left[w_t(\alpha_F)|\mathcal{G}_t\right]|\mathcal{F}_t\right] = \mathbb{E}\left[w_t(\alpha_F)|\mathcal{F}_t\right]. \tag{26}$$

Here the first equality follows from Lemma A.1 and the second from the "law of iterated expectations" (see e.g., Billingsley 1986, p. 470) since $\mathcal{F}_t \subseteq \mathcal{G}_t$. Then, using (26) and Lemma A.1 again, we have

$$\mathbb{E}\left[z_t(\alpha_F)|\mathcal{F}_t\right] = \mathbb{E}\left[z_t^{\mathbb{G}}(\alpha_F)|\mathcal{F}_t\right] - \mathbb{E}\left[z_t^{\mathbb{F}}(\alpha_F)|\mathcal{F}_t\right] = \mathbb{E}\left[w_t(\alpha_F)|\mathcal{F}_t\right] - \mathbb{E}\left[w_t(\alpha_F)|\mathcal{F}_t\right] = 0. \tag{27}$$

This establishes the first part of claim (i). Applying the law of iterated expectations then implies $\mathbb{E}[z_t(\alpha_F)] = \mathbb{E}[\mathbb{E}[z_t(\alpha_F)|\mathcal{F}_t]] = 0$. Summing these these over time implies that $\mathbb{E}[z(\alpha_F)] = 0$, as stated in the second part of claim (i).

(ii) The fact that z is adapted to $\mathbb{G}$ follows from the definition of conditional expectations: for any a, $\mathbb{E}[w_t(a)|\mathcal{G}_t]$ is $\mathcal{G}_t$-measurable and $\mathbb{E}[w_t(a)|\mathcal{F}_t]$ is $\mathcal{F}_t$-measurable and hence $\mathcal{G}_t$-measurable, since $\mathbb{G}$ is a

[1] Billingsley, Patrick (1986). Probability and Measure. John Wiley and Sons, New York.

relaxation of $\mathbb{F}$. Then $z_t(a) = \mathbb{E}[w_t(a)|\mathcal{G}_t] - \mathbb{E}[w_t(a)|\mathcal{F}_t]$ is $\mathcal{G}_t$-measurable. The fact that $z_t(a)$ depends only on the first $t+1$ actions $(a_0, \ldots, a_t)$ of a follows from $w_t(a)$ having this same property. □

A.5. Proof of Theorem 2.3

Proof. The fact that $z^\star$ is dual feasible follows from Proposition 2.2 and the fact that it is optimal for the dual problem follows from the inductive argument in the text preceding the statement of the theorem. The fact that any $\alpha_F^* \in \mathcal{A}_\mathbb{F}$ that is optimal for the primal problem is also optimal for the dual problem then follows by the complementary slackness result, Theorem 2.2.

To establish the last part of the theorem, let us abuse notation a bit and write $V_t(a)$ in place of $V_t(a_0, \ldots, a_t)$ with the understanding that subsequence of actions $(a_0, \ldots, a_t)$ is selected from the full sequence of actions a. If $\alpha_F^* \in \mathcal{A}_\mathbb{F}$ is optimal for the primal problem, we then have

$$\begin{aligned} r(\alpha_F^*) - z^\star(\alpha_F^*) &= \sum_{t=0}^{T} r_t(\alpha_F^*) - z_t^\star(\alpha_F^*) \\ &= \sum_{t=0}^{T} r_t(\alpha_F^*) + \mathbb{E}[V_{t+1}(\alpha_F^*)|\mathcal{F}_t] - \mathbb{E}[V_{t+1}(\alpha_F^*)|\mathcal{G}_t] \\ &= \sum_{t=0}^{T} V_t(\alpha_F^*) - \mathbb{E}[V_{t+1}(\alpha_F^*)|\mathcal{G}_t] \text{ (almost everywhere).} \end{aligned} \tag{28}$$

The first equality is simply the definition of r and $z^\star$ and the second equality follows from Lemma A.1 (using the fact that $\alpha_F^* \in \mathcal{A}_\mathbb{F}$). The final equality follows from the definition of V_t, i.e., we have $V_t(\alpha_F^*) = r_t(\alpha_F^*) + \mathbb{E}[V_{t+1}(\alpha_F^*)|\mathcal{F}_t]$ (almost everywhere); note this equality may fail on a set of measure zero for an optimal policy α_F^*.

If $\mathbb{G}$ is the perfect information relaxation, then $\mathbb{E}[V_{t+1}(\alpha_F^*)|\mathcal{G}_t] = V_{t+1}(\alpha_F^*)$, adjacent terms in (28) cancel and (28) reduces to $V_0(\alpha_F^*) - V_{T+1}(\alpha_F^*)$. Here V_{T+1} was defined defined to be 0 and $V_0(\alpha_F^*)$ is equal to $\mathbb{E}[r(\alpha_F^*)]$. Thus, with a perfect information relaxation, we have $r(\alpha_F^*) - z^\star(\alpha_F^*) = \mathbb{E}[r(\alpha_F^*)]$, almost everywhere. □

A.6. Proof of Proposition 2.3

Proof. (i) Let $z_t^\mathbb{F}(a) = \mathbb{E}[w_t(a)|\mathcal{F}_t]$. The inequality in (12) can be established as follows

$$\begin{aligned} \sup_{\alpha_G \in \mathcal{A}_{\mathbb{G}^1}} \mathbb{E}\left[r(\alpha_G) - z^1(\alpha_G)\right] &= \sup_{\alpha_G \in \mathcal{A}_{\mathbb{G}^1}} \mathbb{E}\left[r(\alpha_G) - \sum_t \left(\mathbb{E}\left[w_t(\alpha_G)|\mathcal{G}_t^1\right] - z_t^\mathbb{F}(\alpha_G)\right)\right] \\ &\leq \sup_{\alpha_G \in \mathcal{A}_{\mathbb{G}^2}} \mathbb{E}\left[r(\alpha_G) - \sum_t \left(\mathbb{E}\left[w_t(\alpha_G)|\mathcal{G}_t^1\right] - z_t^\mathbb{F}(\alpha_G)\right)\right] \\ &= \sup_{\alpha_G \in \mathcal{A}_{\mathbb{G}^2}} \mathbb{E}\left[r(\alpha_G) - \sum_t \left(\mathbb{E}\left[w_t(\alpha_G)|\mathcal{G}_t^2\right] - z_t^\mathbb{F}(\alpha_G)\right)\right] \\ &= \sup_{\alpha_G \in \mathcal{A}_{\mathbb{G}^2}} \mathbb{E}\left[r(\alpha_G) - z^2(\alpha_G)\right] \end{aligned}$$

The first equality follows from the definition of the penalty z^1 and Lemma A.1. The inequality follows from the fact that $\mathcal{A}_{\mathbb{G}^1} \subseteq \mathcal{A}_{\mathbb{G}^2}$ when $\mathbb{G}^2$ is a relaxation of $\mathbb{G}^1$. The next equality follows from the fact that $\mathbb{E}\left[\mathbb{E}\left[w_t(\alpha_G)|\mathcal{G}_t^2\right]\right] = \mathbb{E}\left[\mathbb{E}\left[w_t(\alpha_G)|\mathcal{G}_t^1\right]\right]$ which we will establish shortly. The final equality follows from the definition of the penalty z^2 and Lemma A.1. To see that $\mathbb{E}\left[\mathbb{E}\left[w_t(\alpha_G)|\mathcal{G}_t^2\right]\right] = \mathbb{E}\left[\mathbb{E}\left[w_t(\alpha_G)|\mathcal{G}_t^1\right]\right]$, note that $\mathbb{G}^2$ being a relaxation of $\mathbb{G}^1$ implies that $\mathcal{G}_t^1 \subseteq \mathcal{G}_t^2$. Then, by the law of iterated expectations, we have $\mathbb{E}\left[\mathbb{E}\left[w_t(\alpha_G)|\mathcal{G}_t^2\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[w_t(\alpha_G)|\mathcal{G}_t^2\right]|\mathcal{G}_t^1\right]\right] = \mathbb{E}\left[\mathbb{E}\left[w_t(\alpha_G)|\mathcal{G}_t^1\right]\right]$.

(ii) For any two dual feasible penalties z^1 and z^2 and information relaxation $\mathbb{G}$, we have

$$\begin{aligned} \sup_{\alpha_G \in \mathcal{A}_\mathbb{G}} \mathbb{E}\left[r(\alpha_G) - z^1(\alpha_G)\right] &= \sup_{\alpha_G \in \mathcal{A}_\mathbb{G}} \mathbb{E}\left[r(\alpha_G) - z^2(\alpha_G) + (z^2(\alpha_G) - z^1(\alpha_G))\right] && (29) \\ &\leq \sup_{\alpha_G \in \mathcal{A}_\mathbb{G}} \mathbb{E}\left[r(\alpha_G) - z^2(\alpha_G)\right] + \sup_{\alpha_G \in \mathcal{A}_\mathbb{G}} \mathbb{E}\left[z^2(\alpha_G) - z^1(\alpha_G)\right] && (30) \end{aligned}$$

Rearranging this yields the inequality on the right in (13). Interchanging z^1 and z^2 and multiplying through by -1 yields the inequality on the left in (13).

(iii) The proof here follows the proof of Proposition 2.2, except we now take $z_t^{\mathbb{F}}(a) = \mathbb{E}\left[w_t(a)|\mathcal{F}'_t\right]$ and must show that $\mathbb{E}\left[z_t^{\mathbb{F}}(\alpha_F)|\mathcal{F}_t\right] = \mathbb{E}\left[w_t(\alpha_F)|\mathcal{F}_t\right]$ for any $\mathbb{F}$-adapted α_F. This can be established as follows:

$$\mathbb{E}\left[z_t^{\mathbb{F}}(\alpha_F)|\mathcal{F}_t\right] = \mathbb{E}\left[\mathbb{E}\left[w_t(\alpha_F)|\mathcal{F}'_t\right]|\mathcal{F}_t\right] = \mathbb{E}\left[w_t(\alpha_F)|\mathcal{F}_t\right]. \tag{31}$$

Here the first equality follows from Lemma A.1 (since α_F being $\mathbb{F}$-adapted implies α_F is also $\mathbb{F}'$-adapted) and the second equality follows from the law of iterated expectations since $\mathcal{F}_t \subseteq \mathcal{G}_t$. The rest of the proof then proceeds as in the proof of Proposition 2.2.

(iv) Suppose $\alpha_G^* \in \mathcal{A}_{\mathbb{G}}$ is an optimal solution for the left side of (14). Then we have

$$\begin{aligned}
\sup_{\alpha_G \in \mathcal{A}_{\hat{\mathbb{G}}}} \mathbb{E}\left[r(\alpha_G) - \hat{z}(\alpha_G)\right] &\geq \mathbb{E}\left[r(\alpha_G^*) - \hat{z}(\alpha_G^*)\right] \\
&= \mathbb{E}\left[r(\alpha_G^*) - \sum_{t=0}^{T} \hat{z}_t(\alpha_G^*)\right] \\
&= \mathbb{E}\left[r(\alpha_G^*) - \sum_{t=0}^{T} \mathbb{E}\left[\hat{z}_t(\alpha_G^*)|\mathcal{G}_t\right]\right] \\
&= \mathbb{E}\left[r(\alpha_G^*) - \sum_{t=0}^{T} z_t(a)\right] \\
&= \mathbb{E}\left[r(\alpha_G^*) - z(\alpha_G^*)\right] \\
&= \sup_{\alpha_G \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}\left[r(\alpha_G) - z(\alpha_G)\right]
\end{aligned}$$

The first inequality follows from the fact that $\alpha_G^* \in \mathcal{A}_{\hat{\mathbb{G}}}$ (since $\mathbb{G} \subseteq \hat{\mathbb{G}}$). The next two equalities follow from the definition of $\hat{z}$ and the law of iterated expectations, respectively. The third equality follows from the estimate being unbiased and Lemma A.1: the estimate being unbiased means $z_t(a) = \mathbb{E}\left[\hat{z}_t(a)|\mathcal{G}_t\right]$ and, since α_G^* is $\mathbb{G}$-adapted, Lemma A.1 implies $\mathbb{E}\left[z_t(\alpha_G^*)\right] = \mathbb{E}\left[\hat{z}_t(\alpha_G^*)|\mathcal{G}_t\right]$. The fourth equality follows from the definition of z and the final equality follows from the definition of α_G^* as an optimal solution for the left side of (14). If there is no α_G^* that attains the optimal value (i.e., the supremum is approached but not attained), for any $\epsilon > 0$ we can find an $\alpha_G^* \in \mathcal{A}_{\mathbb{G}}$ that is within ϵ of the optimal value. The argument above goes through for this α_G^* except the last line becomes

$$\geq \sup_{\alpha_G \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}\left[r(\alpha_G) - z(\alpha_G)\right] - \epsilon.$$

Because ϵ can be made arbitrarily small, the desired result still holds. □

A.7. Imperfect Information Bounds for the Adaptive Inventory Example

As noted in §3.4, in this case, we would generate demands $\hat{d}_t^k$ and the corresponding distributions $\hat{\pi}_t^k$ in the kth trial of the outer simulation. The inner dual problem is then a stochastic DP that explicitly considers the uncertainty about the ordering costs: the lower-bound cost-to-go function $\underline{\mathcal{J}}_t^{L,k}(x_t, c_t)$ in the kth trial can be written recursively as

$$\begin{aligned}
\underline{\mathcal{J}}_t^{L,k}(x_t, c_t) &= -c_t x_t + \min_{y_t \geq x_t} \Big\{ c_t y_t + \mathbb{E}\left[\underline{\mathcal{J}}_{t+1}^{L,k}(y_t - \hat{d}_t^k, \tilde{c}_{t+1}) - J_{t+1}^{L-1}(y_t - \hat{d}_t^k; \tilde{c}_{t+1}, \hat{\pi}_{t+1}^k)|c_t\right] \\
&\quad + \mathbb{E}\left[f_t(y_t - \tilde{d}_t) + J_{t+1}^{L-1}(y_t - \tilde{d}_t; \tilde{c}_{t+1}, \pi_{t+1}(\hat{\pi}_t^k, \tilde{d}_t))|\hat{\pi}_t^k, c_t\right] \Big\}
\end{aligned} \tag{32}$$

with the terminal value $\underline{\mathcal{J}}_T^{L,k}(x_T, c_T) = -c_T x_T$.

A.8. Derivation of Equation (22)

Using Ito's lemma and equation (19), we can write the diffusion equation for $S_\tau = ln(s_\tau)$ as

$$dS_\tau = (\gamma_\tau - \frac{1}{2}v_\tau)d\tau + \sqrt{v_\tau}dz^s_\tau.$$

Using a discrete-time approximation of this diffusion equation with time steps of length δ and taking $\mathcal{G}_t$ to represent the state of information where all interest rates and volatilities are known, S_{t+1} is normally distributed with mean and variance:

$$\begin{aligned}\mathbb{E}\left[S_{t+1}|\mathcal{G}_t\right] &= S_t + (\gamma_t - \frac{1}{2}v_t)\delta + \rho_{sv}\sqrt{v_t}(v_{t+1} - v_t)\\ \text{Var}\left[S_{t+1}|\mathcal{G}_t\right] &= (1-\rho_{sv}^2)v_t\delta\end{aligned}$$

The stock price $s_t = \exp(S_t)$ is then log-normally distributed with mean

$$\mathbb{E}\left[S_{t+1}|\mathcal{G}_t\right] = \exp\left(\mathbb{E}\left[S_{t+1}|\mathcal{G}_t\right] + \frac{1}{2}\text{Var}\left[S_{t+1}|\mathcal{G}_t\right]\right)$$

Equation (22) then follows by substituting the expressions above for the mean and variance of S_{t+1}.

A.9. Comparison of Option Pricing Bounds with Haugh and Kogan (2004)

As noted in the text, with the perfect information relaxation the "flattened" version of the inner problem for the option pricing example (see equation (24)) is very close to the form of martingale-based duality considered by Haugh and Kogan (2004), Rogers (2002), and Andersen and Broadie (2004), but there is a subtle difference. Taking expectations over equation (24), our upper bound on the value of a call option is given by:

$$\mathbb{E}\left[\overline{v}_0\right] = \mathbb{E}\left[\max\left\{\max_{t\in\{0,\ldots,T\}}\left\{\phi_t\ (s_t - K) - \mu_t\right\}, -\mu_T\right\}\right] \tag{33}$$

where μ is martingale with $\mu_0 = 0$. Haugh and Kogan (2004), Rogers (2002), and Andersen and Broadie (2004) consider general martingales (i.e., μ_0 need not be 0) and write the dual upper bound on the option as

$$\mathbb{E}\left[\max_{t\in\{0,\ldots,T\}}\left\{\phi_t h_t - \mu_t\right\}\right] + \mu_0 \tag{34}$$

where h_t is the option payoff function. Including the μ_0 term here allows the use of martingales with non-zero initial values, but this is not a substantive difference in formulations: We could always replace μ_t with $\mu_t - \mu_0$ in (34) and have the same bound but with $\mu_0 = 0$. Alternatively, we could add μ_0 to (33).

The subtle difference centers on the definition of the option payoff function h_t and how non-exercise is handled. Haugh and Kogan and others require h_t to be a non-negative function that describes the payoffs of the option if exercised in period t. For a call option, they take $h_t = \max\{0, (s_t - K)\}$. There is a small abuse of terminology here: we do not "exercise" an option to get a 0 payoff. We could, however, perhaps throw away or "burn" an option. The possibility of burning an option before expiration doesn't matter in the primal problem, because we would never burn an option before it expires.

However, the possibility of burning an option may matter in the dual. Compare the maximization problems in (33) and (34) in the case of a call option. Problem (34) allows the DM to burn the call option before expiration in period t ($t < T$) and receive $-\mu_t$ or exercise the option and receive $\phi_t(s_t - K) - \mu_t$. In (33), the DM can receive $\phi_t(s_t - K) - \mu_t$, but cannot receive $-\mu_t$ alone. In other words, for a call option, we have

$$\max_t\left\{\phi_t h_t - \mu_t\right\} = \max_t\left\{\phi_t \max\{0, (s_t - K)\} - \mu_t\right\} \geq \max\left\{\max_t\left\{\phi_t(s_t - K) - \mu_t\right\}, -\mu_T\right\} \tag{35}$$

and it could be the case that the inequality is strict for some μ_t. Thus the two formulations (33) and (34) are slightly different, with (33) providing tighter bounds.

B. Comparison with Stochastic Programming Duality Results

As mentioned in the introduction, the idea of relaxing the nonanticipativity constraints has been exploited in the stochastic programming (SP) literature. In the SP literature, the nonanticipativity constraints discussed in our paper are sometimes called "implementability" constraints. Here we briefly review this SP formulation and compare it to ours. We also briefly compare our formulation to that of Rogers (2007).

B.1. Stochastic Programming Duality

Our description of the SP approach follows Shapiro and Ruszczynski (2007, pp. 55-75; hereafter SR) and follows them in focusing on perfect information relaxations. The first main assumption is to assume the actions a_t are scalars or, more generally, vectors in $\mathbb{R}^n$. Let $\alpha_t(\omega)$ denote the action selected in period t by policy α with outcome ω. The nonanticipativity constraints require the DM to select the same period-t action in all outcomes that are indistinguishable at time t. We can write these constraints as $\alpha_t = \mathbb{E}[\alpha_t|\mathcal{F}_t]$, so the selected action, viewed as a random variable, is equal to its own expected value conditional on the time-t state of information.

SR then place some assumptions on the reward functions and action spaces and use standard Lagrange duality arguments to "dualize" the nonanticipativity constraint. First, assume the reward functions $(r_0(a,\omega),\ldots,r_T(a,\omega))$ depend on the action selected in period t but are independent of the actions selected in other periods. Second, assume that each $r_t(a,\omega)$ is polyhedral (piecewise linear and convex) in a_t for all ω. Third, assume the sequences of actions $a = (a_0,\ldots,a_T)$ are drawn from a $A \subseteq \mathbb{R}^{n(T+1)}$ defined by a set of linear constraints. Now, let λ_t be the Lagrange multipliers associated with the nonanticipativity constraints requiring $\alpha_t = \mathbb{E}[\alpha_t|\mathcal{F}_t]$. Since α_t is a random variable, λ_t is also a random variable (i.e., a function of ω) and has the same dimensionality as α_t. SR then show that the Lagrangian dual of the stochastic program can be written as

$$\min_{\{\lambda_t:\mathbb{E}[\lambda_t|\mathcal{F}_t]=0\}} \mathbb{E}\left[\max_{a\in A}\left\{\sum_{t=0}^{T} r_t(a_t)+\lambda_t a_t\right\}\right]. \tag{36}$$

Standard Lagrange duality arguments ensure that weak and strong duality hold in this framework.

In (36), the $\lambda_t\alpha_t$ term in the objective function can be viewed as analogous to a linear penalty. The constraint $\mathbb{E}[\lambda_t|\mathcal{F}_t] = 0$ is equivalent to requiring $\mathbb{E}[\alpha_t\lambda_t|\mathcal{F}_t] = 0$ for all nonanticipative α_t, which is analogous to our definition of dual feasible penalties (3), albeit with an equality constraint in place of the inequality. The optimization in (36) is no longer constrained by the nonanticipativity constraint and is analogous to the inner problem for the perfect information relaxation in equation (5) above. This allows us to decompose the inner problem into a series of scenario-specific subproblems for a given set of Lagrange multipliers.

Our formulation of the primal DP problem (1) is more general than the SP formulation in that we do not place any restrictions on the action spaces or reward functions and, in the dual, we do not require the penalties to be linear functions of the actions; weak and strong duality hold without these assumptions. As formulated, our examples do not fit within the SP formulation. The option pricing example has a discrete action space. The inventory model has integer constraints on the order quantities and, if even if we ignore these integer constraints, the penalties we considered in the inventory example are not linear functions of the actions and hence are not consistent with the SP formulation.

B.2. Linear Programming Formulation of DP Duality

As discussed in §2.2, there are strong connections between our results and standard results in linear programming. In fact, if we allow the use of mixed policies, then our formulation of the primal DP can be viewed as a linear programming problem where the decision variables are mixing probabilities on policies; the objective function and constraints are both linear functions of the mixing probabilities. Applying Lagrange duality arguments like those used in the SP framework in our linear programming formulation of the primal delivers results and penalties like ours. However, as shown in §2.2, we can also use simple, direct arguments to establish the duality results. In this subsection, we describe this linear programming formulation and duality argument. For ease of exposition, we will assume that the set of outcomes Ω and actions sequences A are finite sets and, hence, the set of all policies $\mathcal{A}$ is finite as well, with $|\mathcal{A}| = |\Omega|^{|A|}$.

In our "mixed" version of the primal problem (1), rather than selecting a policy α that selects an action sequence a in given outcome ω (i.e., $\alpha : \Omega \to A$), we instead randomly choose a policy $\alpha \in \mathcal{A}$

using a probability measure μ. A mixed policy μ is nonanticipative if its mass is concentrated on the set of nonanticipative policies $\mathcal{A}_{\mathbb{F}}$. Let M and $M_{\mathbb{F}}$ be the set of all mixed policies and all nonanticipative mixed policies, respectively. Clearly M is a convex set with extreme points corresponding to degenerate distributions that place all of their mass on a single policy α. Similarly, $M_{\mathbb{F}}$ is a convex set with extreme points corresponding to degenerate distributions that place all of their mass on a single nonanticipative policy α.

The mixed version of the original primal (1) can be written as

$$\max_{\mu \in M_{\mathbb{F}}} \mathbb{E}\left[\mu' \rho\right], \tag{37}$$

where $\rho(\omega) = (r(\alpha(\omega), \omega))_{\alpha \in \mathcal{A}}$ is a random vector describing the rewards for each policy α. (Note $\rho : \Omega \to \mathbb{R}^{|\mathcal{A}|}$.) The inner product $\mu'\rho$ is a random variable (a function of ω) that represents the expected rewards associated with the mixed policy μ, with the expectations taken over the mixture of policies, not the outcomes ω. Note the objective function is linear in μ and the constraint set $M_{\mathbb{F}}$ is convex in μ; thus the optimal value will be attained at an extreme point of $M_{\mathbb{F}}$. As noted above, the extreme points of $M_{\mathbb{F}}$ correspond to the degenerate mixed policies that place all of their mass on a single nonanticipative policy α. Thus the optimal value for the mixed primal (37) will match that of the original primal (1) and each optimal solution for the mixed primal will correspond to a nonanticipative policy that is optimal for the original problem or perhaps a mixture of nonanticipative policies, each of which is optimal for the original problem.

Next we develop a linear equality based representation of the nonanticipativity constraint. First note that we can define nonanticipativity for non-mixed policies using an indicator function $1_{a_0,\ldots,a_t}(a)$ on A that takes on the value 1 if the first t elements of a match $a_0, \ldots, a_t$ and is zero otherwise; $1_{a_0,\ldots,a_t}(\alpha)$ is a random variable (a function of ω) that takes on 1 when α selects the sequence $(a_0, \ldots, a_t)$ and is zero otherwise. A policy α is nonanticipative if and only if

$$1_{a_0,\ldots,a_t}(\alpha) = \mathbb{E}\left[1_{a_0,\ldots,a_t}(\alpha)|\mathcal{F}_t\right] \quad \text{for all } t \text{ and } a_0, \ldots, a_t. \tag{38}$$

Note that both sides of (38) are random variables and the equality constraints must hold for every ω. We now generalize this idea to mixed policies. Let $\mu_t(a_0, \ldots, a_t; \omega)$ denote the probability of choosing the action (sub)sequence $(a_0, \ldots, a_t)$ given outcome ω and mixed policy μ. This probability $\mu_t(a_0, \ldots, a_t; \omega)$ can be calculated as the inner product $\mu' \mathbb{1}_{a_0,\ldots,a_t}(\omega)$ where $\mathbb{1}_{a_0,\ldots,a_t}(\omega) = (1_{a_0,\ldots,a_t}(\alpha(\omega)))_{\alpha \in \mathcal{A}}$. Here $\mathbb{1}_{a_0,\ldots,a_t}(\omega)$ is a random vector with entries noting whether policy α matches the specified action subsequence for the given outcome ω. (Note $\mathbb{1}_{a_0,\ldots,a_t} : \Omega \to \{0,1\}^{|\mathcal{A}|}$.) Suppressing the outcome ω, we can view $\mu_t(a_0, \ldots, a_t) = \mu' \mathbb{1}_{a_0,\ldots,a_t}$ as a random variable. Using this, we can write a linear constraint that requires the probability of choosing an action sequence $(a_0, \ldots, a_t)$ under μ to be $\mathcal{F}_t$-measurable:

$$\mu_t(a_0, \ldots, a_t) = \mathbb{E}\left[\mu_t(a_0, \ldots, a_t)|\mathcal{F}_t\right] \quad \text{for all } t \text{ and } (a_0, \ldots, a_t). \tag{39}$$

If this condition is satisfied, we can build a "decision tree" to describe the expected payoffs of the problem with well-defined probabilities for each decision node that are conditioned on the period-t state of information $\mathcal{F}_t$ and the prior history of actions. The conditional probabilities for period-t are given by $\mu_t(a_0, \ldots, a_t)/\mu_{t-1}(a_0, \ldots, a_{t-1})$ and (39) ensures that these conditional probabilities are measurable with respect to $\mathcal{F}_t$, so that they depend only on the outcomes of uncertainties that have already been resolved (e.g., on chance nodes that appear before the decisions in the decision tree).

We can now consider an alternative version of the mixed primal (37) with the linear constraint (39) replacing the nonanticipativity constraint ($\mu \in M_{\mathbb{F}}$):

$$\begin{array}{rc} \max_{\mu \in M} & \mathbb{E}\left[\mu'\rho\right] \\ \text{s.t.} & \mu_t(a_0, \ldots, a_t) = \mathbb{E}\left[\mu_t(a_0, \ldots, a_t)|\mathcal{F}_t\right] \quad \text{for all } t \text{ and } (a_0, \ldots, a_t). \end{array} \tag{40}$$

It is straightforward to show that any nonanticipative mixture satisfies (39): the nonanticipative mixtures assign positive probability only to policies that satisfy (38) and thus the nonanticipative mixtures must satisfy (39). The constraint (39) may also be satisfied by mixed policies μ that are not nonanticipative, so (40) is a relaxation of (37). However, for any mixed policy μ satisfying (39), we can construct a nonanticipative mixed policy that is "behaviorally equivalent" to μ in that it leads to the same joint probability distribution on action sequences and outcomes ($A \times \Omega$) and thus leads to the same expected rewards; this follows from a famous result in game theory known as Kuhn's Theorem (see, e.g., Fudenberg and Tirole, 1991).[2] Given this, replacing the nonanticipativity constraint in (37) with the relaxed constraint (39) does not improve the optimal value and we can construct a behaviorally equivalent nonanticipative mixed policy corresponding to any solution to (40).

Having established that the original primal (1) and linear mixed primal (40) have equal optimal values and corresponding solutions, we now proceed to consider the Lagrangian dual of the mixed primal (40) by relaxing the constraints forcing the mixed policies μ to satisfy the linear constraints (39). The dual function can be written

$$g(w_t) = \max_{\mu \in M} \mathbb{E}\left[\rho'\mu - \sum_{t=0}^{T} w_t'(\mu_t - \mathbb{E}[\mu_t|\mathcal{F}_t])\right]. \tag{41}$$

The Lagrange multipliers associated with the constraints (39) are given by a stochastic process $w_t(\omega) \in \mathbb{R}^{|A_t|}$ where A_t is the set of all possible subsequences of actions $(a_0, \ldots, a_t)$. (Note that these are the usual Lagrange multipliers divided by $\mathbb{P}(\{\omega\})$, so we can bring the Lagrange multipliers inside the expectation in (41).)

Recall that for any $\mathcal{F}$-measurable random variable X and any random variable Y, we have $X\mathbb{E}[Y|\mathcal{F}] = \mathbb{E}[XY|\mathcal{F}]$. Noting this several times and using iterated expectations, we observe:

$$\begin{aligned}
\mathbb{E}[w_t'(\mu_t - \mathbb{E}[\mu_t|\mathcal{F}_t])] &= \mathbb{E}[\mathbb{E}[w_t'(\mu_t - \mathbb{E}[\mu_t|\mathcal{F}_t])|\mathcal{F}_t]] \\
&= \mathbb{E}[\mathbb{E}[w_t'\mu_t|\mathcal{F}_t] - \mathbb{E}[w_t'\mathbb{E}[\mu_t|\mathcal{F}_t]|\mathcal{F}_t]] \\
&= \mathbb{E}\left[\mathbb{E}[w_t'\mu_t|\mathcal{F}_t] - \mathbb{E}[w_t|\mathcal{F}_t]'\,\mathbb{E}[\mu_t|\mathcal{F}_t]\right] \\
&= \mathbb{E}\left[\mathbb{E}[w_t'\mu_t|\mathcal{F}_t] - \mathbb{E}\left[\mathbb{E}[w_t|\mathcal{F}_t]'\,\mu_t|\mathcal{F}_t\right]\right] \\
&= \mathbb{E}[\mathbb{E}[\mu_t'(w_t - \mathbb{E}[w_t|\mathcal{F}_t])|\mathcal{F}_t]] \\
&= \mathbb{E}[\mathbb{E}[\mu_t'z_t|\mathcal{F}_t]] \\
&= \mathbb{E}[\mu_t'z_t],
\end{aligned}$$

where $z_t = w_t - \mathbb{E}[w_t|\mathcal{F}_t]$ and, by construction, we have $\mathbb{E}[z_t|\mathcal{F}_t] = 0$. These period-$t$ penalties z_t are thus constructed like our good penalties as the expectations of the generating functions w_t that depend on the actions from the first t periods (and ω). Using this, we can rewrite the dual function (41) as

$$\max_{\mu \in M} \mathbb{E}\left[\rho'\mu - \sum_{t=0}^{T} z_t'\mu_t\right]. \tag{42}$$

Any dual feasible z_t, that is, any z_t satisfying $\mathbb{E}[z_t|\mathcal{F}_t] = 0$ (or any set of generating functions w_t that depend on the first t-periods actions and ω) will generate an upper bound on the mixed primal problem (40) and hence the original primal (1). Strong Lagrangian duality implies that there exists a z_t such that the bounds are tight.

Recalling that $\mu_t = \mu'\mathbb{1}_{a_0,\ldots,a_t}$, we see that this Lagrangian is linear in μ and obtains the maximum in (42) at an extreme point of M which concentrates all of its mass at a single policy α in $\mathcal{A}$. Thus (42) can be written as

$$\max_{\alpha \in \mathcal{A}} \mathbb{E}\left[r(\alpha) - \sum_{t=0}^{T} z_t(\alpha)\right]. \tag{43}$$

With no constraints on the policy chosen, we can decompose this into a series of outcome-specific optimization problems where we choose the action a for each ω and rewrite (43) as

$$\mathbb{E}\left[\max_{a \in A}\left\{r(a) - \sum_{t=0}^{T} z_t(a)\right\}\right]. \tag{44}$$

[2] Fudenberg, Drew and Tirole, Jean (1991). *Game Theory.* The MIT Press, Cambridge, Massachusetts.

Thus, for any dual feasible penalty $(z_0, \ldots, z_T)$ (or Lagrange multipliers/generating functions $(w_0, \ldots, w_T)$), (44) generates an upper bound on the original primal (1) and there exist a penalty/generating function that leads to a tight upper bound. This is exactly the perfect information bound given in equation (5) above.

B.3. Relationship to Rogers (2007)

Rogers (2007) recently independently proposed an extension of his dual approach to option pricing (Rogers 2002) to Markov decision processes. He considers only the perfect information relaxations and assumes the DP has a Markovian structure with a period-t state variable X_t that, in our notation, can be viewed as a function of ω and the action vector a. Rogers considers period-t "penalties" of the form $\mathbb{E}[h_{t+1}(X_{t+1})|\mathcal{F}_t] - h_{t+1}(X_{t+1})$. These penalties are similar to those generated by our Proposition 2.2 except his generating functions h_t depend on the state X_t alone whereas our generating functions w_t can depend on both the outcome ω and actions a.

Rogers shows that weak and strong duality holds with penalties of this form; strong duality is established by taking $h_t(X_t)$ to be the dynamic programming value function. Rogers provides some ideas and results towards constructing an algorithm for approximately solving a Markov decision problem, but does not consider any specific applications of the approach or numerical examples. Our approach is more general than Rogers in that we do not require the DP to have a Markov structure, we consider imperfect as well as perfect information relaxations, we consider a larger class of penalties, and we present many additional results (e.g., complementary slackness, properties of penalties and relaxations) and some specific examples.

C. Further Details on the Adaptive Inventory Example

This appendix provides the detailed assumptions for the inventory example that were omitted from the main body of the paper. The seven different state transition matrices are described in Table 5 and the four different prior distributions are shown in Table 6. Figure 2 shows the transition probabilities for the Markov chain for the ordering costs c_t. Figure 3 shows the lower bounds generated by using the perfect information relaxation with the zero-, one- and two-period look-ahead penalties. The format is the same as the "aquarium plot" of Figure 1. Tables 7 and 8 provide the values and mean standard errors underlying Figure 1 and Figure 3. Table 9 provides the results for the modified myopic policy discussed in §3.6.

Table 5: Transition Probability Matrices.

		Demand distribution (δ_t)		
		Low	Medium	High
	Low	0.8	0.1	0.1
Stable, Pos. Corr.	Medium	0.1	0.8	0.1
	High	0.1	0.1	0.8
	Low	0.2	0.4	0.4
Stable, Neg. Corr.	Medium	0.4	0.2	0.4
	High	0.4	0.4	0.2
	Low	0.333	0.333	0.333
Stable, Zero Corr.	Medium	0.333	0.333	0.333
	High	0.333	0.333	0.333
	Low	0.8	0.1	0.1
Upward, Slow	Medium	0.0	0.8	0.2
	High	0.0	0.0	1.0
	Low	0.2	0.4	0.4
Upward, Fast	Medium	0.0	0.2	0.8
	High	0.0	0.0	1.0
	Low	1.0	0.0	0.0
Downward, Slow	Medium	0.2	0.8	0.0
	High	0.1	0.1	0.8
	Low	1.0	0.0	0.0
Downward, Fast	Medium	0.8	0.2	0.0
	High	0.4	0.4	0.2

Table 6: Priors.

	Demand distribution (δ_t)		
	Low	Medium	High
H: High	0.10	0.30	0.60
U: Uniform	0.33	0.34	0.33
M: Medium	0.10	0.80	0.10
L: Low	0.60	0.30	0.10

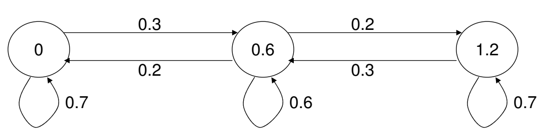

Figure 2: Dynamics of c_t.

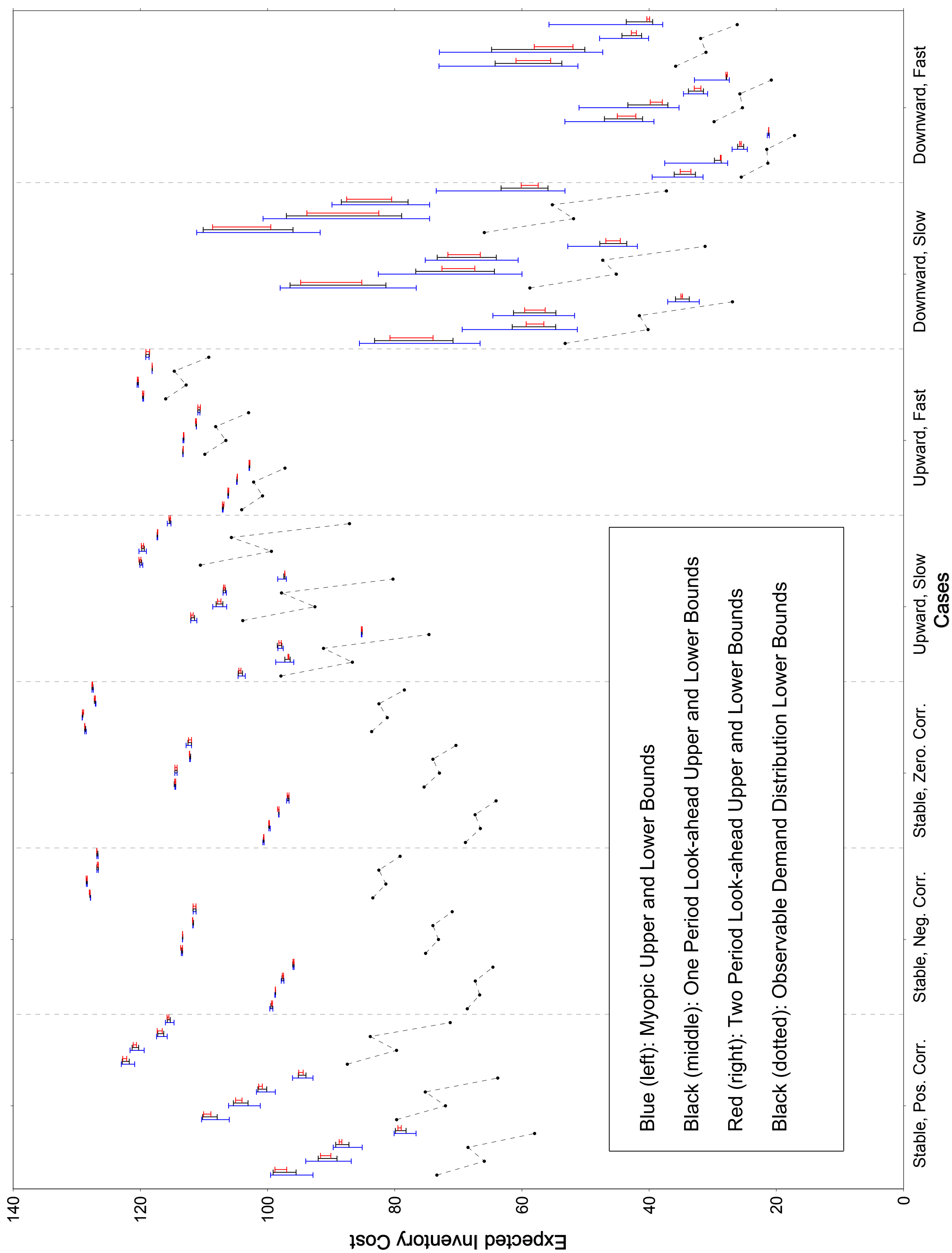

Figure 3: Upper and lower bounds with the imperfect information relaxation.

Perfect Information Relaxation

Transition Matrix	Shortage Penalty	Prior	Myopic Lower	MSE	Upper	MSE	Gap	MSE	One-period look-ahead Lower	MSE	Upper	MSE	Gap	MSE	Two-period look-ahead Lower	MSE	Upper	MSE	Gap	MSE
Stable, Positive Correlation	Low (1)	High	92.999	0.868	99.712	0.958	6.714	0.306	95.538	0.555	99.279	0.956	3.741	0.477	97.031	0.352	98.992	0.950	1.961	0.640
		Uniform	86.682	0.820	93.752	0.934	7.070	0.297	88.996	0.543	91.917	0.978	2.922	0.494	90.040	0.359	91.515	1.000	1.476	0.680
		Medium	84.419	0.795	88.949	0.916	4.530	0.285	86.714	0.511	88.441	0.914	1.727	0.465	87.988	0.324	87.858	0.904	-0.130	0.616
		Low	75.255	0.784	78.696	0.995	3.440	0.333	77.478	0.530	78.439	0.995	0.961	0.518	78.600	0.356	78.217	0.988	-0.383	0.669
	Med. (1.85)	High	105.450	0.869	109.711	0.923	4.261	0.289	107.490	0.552	109.633	0.928	2.143	0.449	108.594	0.349	109.482	0.908	0.888	0.599
		Uniform	100.191	0.827	104.991	0.900	4.799	0.280	102.468	0.536	104.128	0.918	1.660	0.450	103.610	0.348	103.904	0.919	0.294	0.611
		Medium	98.470	0.809	101.368	0.900	2.898	0.293	99.850	0.521	101.139	0.888	1.290	0.436	100.607	0.330	100.882	0.876	0.275	0.586
		Low	92.831	0.823	96.142	0.997	3.311	0.313	94.018	0.549	95.117	1.013	1.098	0.516	94.521	0.360	95.028	1.004	0.508	0.680
	High (4)	High	122.484	0.875	124.585	0.865	2.101	0.337	122.713	0.552	124.312	0.856	1.599	0.410	122.731	0.347	124.278	0.844	1.547	0.550
		Uniform	118.468	0.855	120.700	0.865	2.232	0.343	119.788	0.541	120.439	0.865	0.651	0.426	120.309	0.338	120.215	0.846	-0.095	0.562
		Medium	114.567	0.859	116.596	0.882	2.030	0.317	115.526	0.545	116.510	0.871	0.984	0.415	115.974	0.341	116.692	0.861	0.718	0.562
		Low	115.187	0.765	116.637	0.828	1.450	0.353	115.638	0.495	116.283	0.827	0.645	0.438	115.775	0.317	116.582	0.823	0.807	0.565
Stable, Negative Correlation	Low (1)	High	99.165	0.686	99.365	0.745	0.201	0.401	99.171	0.440	99.260	0.740	0.089	0.402	99.179	0.280	99.187	0.721	0.007	0.499
		Uniform	98.866	0.677	98.894	0.768	0.028	0.411	98.914	0.434	98.726	0.762	-0.187	0.430	98.859	0.274	98.784	0.750	-0.075	0.532
		Medium	96.984	0.675	97.351	0.762	0.367	0.390	97.144	0.434	97.190	0.753	0.046	0.408	97.228	0.273	97.101	0.741	-0.127	0.517
		Low	95.762	0.692	95.569	0.779	-0.193	0.405	95.679	0.439	95.651	0.771	-0.029	0.423	95.698	0.279	95.645	0.759	-0.054	0.530
	Med. (1.85)	High	113.921	0.750	113.781	0.753	-0.140	0.428	113.596	0.482	113.685	0.739	0.089	0.390	113.438	0.306	113.938	0.732	0.500	0.493
		Uniform	113.542	0.739	113.496	0.749	-0.047	0.442	113.515	0.478	113.512	0.750	-0.003	0.409	113.418	0.303	113.618	0.745	0.200	0.508
		Medium	112.728	0.772	112.278	0.775	-0.450	0.435	112.312	0.495	112.186	0.762	-0.126	0.407	112.011	0.315	112.428	0.753	0.417	0.503
		Low	111.508	0.782	111.872	0.809	0.363	0.433	111.433	0.501	111.886	0.792	0.453	0.412	111.389	0.318	111.795	0.779	0.406	0.522
	High (4)	High	125.957	0.824	126.397	0.741	0.440	0.441	126.866	0.530	126.478	0.731	-0.388	0.369	127.385	0.337	126.550	0.718	-0.835	0.457
		Uniform	127.682	0.825	127.537	0.746	-0.144	0.455	127.962	0.528	127.660	0.746	-0.301	0.391	128.218	0.333	127.669	0.731	-0.549	0.478
		Medium	126.980	0.793	126.917	0.739	-0.063	0.436	126.822	0.513	126.930	0.725	0.107	0.375	126.710	0.326	127.016	0.715	0.305	0.462
		Low	127.133	0.837	127.109	0.759	-0.024	0.455	127.101	0.539	127.086	0.751	-0.015	0.393	126.967	0.341	127.227	0.744	0.260	0.484
Stable, Zero Correlation	Low (1)	High	100.498	0.694	100.691	0.788	0.194	0.412	100.545	0.443	100.555	0.772	0.009	0.451	100.564	0.278	100.462	0.751	-0.102	0.534
		Uniform	100.799	0.692	100.470	0.786	-0.329	0.414	100.434	0.443	100.598	0.780	0.164	0.460	100.173	0.282	100.542	0.755	0.369	0.537
		Medium	98.268	0.671	97.932	0.764	-0.336	0.411	98.223	0.432	97.914	0.744	-0.309	0.440	98.170	0.275	98.223	0.731	0.054	0.517
		Low	98.091	0.679	97.997	0.809	-0.093	0.421	97.520	0.438	97.967	0.795	0.447	0.473	97.115	0.279	98.169	0.783	1.054	0.564
	Med. (1.85)	High	113.784	0.753	113.834	0.782	0.050	0.446	113.932	0.486	113.935	0.769	0.003	0.438	114.086	0.308	113.920	0.753	-0.166	0.517
		Uniform	113.013	0.760	113.958	0.811	0.945	0.421	113.455	0.485	113.888	0.799	0.433	0.451	113.743	0.305	113.784	0.777	0.042	0.537
		Medium	112.616	0.749	112.576	0.781	-0.040	0.410	112.380	0.485	112.582	0.773	0.202	0.425	112.246	0.306	112.632	0.764	0.386	0.522
		Low	110.885	0.753	111.828	0.793	0.944	0.436	111.241	0.485	111.538	0.784	0.297	0.454	111.466	0.305	111.449	0.767	-0.016	0.533
	High (4)	High	128.820	0.826	128.661	0.775	-0.159	0.467	128.812	0.533	128.833	0.757	0.021	0.410	128.786	0.339	128.862	0.743	0.075	0.484
		Uniform	128.344	0.833	128.572	0.767	0.228	0.445	128.683	0.534	128.556	0.763	-0.127	0.408	128.879	0.336	128.578	0.747	-0.302	0.489
		Medium	127.986	0.821	127.425	0.782	-0.561	0.447	127.720	0.527	127.758	0.777	0.039	0.424	127.520	0.337	127.858	0.764	0.338	0.510
		Low	127.772	0.825	126.943	0.734	-0.829	0.447	127.660	0.535	127.080	0.727	-0.580	0.386	127.519	0.338	127.252	0.715	-0.267	0.460
Upward, Slow	Low (1)	High	101.797	1.131	102.984	1.147	1.187	0.235	102.796	0.736	102.953	1.141	0.157	0.472	103.532	0.475	102.723	1.138	-0.809	0.708
		Uniform	96.151	1.115	99.187	1.160	3.037	0.236	96.531	0.750	97.642	1.196	1.111	0.509	96.712	0.502	97.040	1.214	0.327	0.760
		Medium	96.926	1.067	97.768	1.116	0.842	0.231	97.212	0.697	97.789	1.118	0.577	0.481	97.416	0.451	97.392	1.114	-0.024	0.708
		Low	85.483	1.091	85.756	1.239	0.273	0.274	85.018	0.742	85.547	1.242	0.529	0.552	84.947	0.504	85.243	1.222	0.296	0.762
	Med. (1.85)	High	109.740	1.147	110.701	1.128	0.961	0.235	110.559	0.743	110.603	1.125	0.044	0.454	111.206	0.478	110.599	1.111	-0.607	0.683
		Uniform	104.732	1.099	107.134	1.103	2.402	0.230	105.883	0.728	106.569	1.116	0.686	0.454	106.604	0.476	106.377	1.120	-0.227	0.691
		Medium	106.292	1.081	106.762	1.081	0.469	0.250	106.267	0.707	106.711	1.080	0.445	0.451	106.382	0.456	106.439	1.070	0.057	0.666
		Low	97.210	1.019	98.634	1.097	1.424	0.258	97.469	0.692	97.599	1.125	0.130	0.496	97.504	0.464	97.557	1.123	0.053	0.705
	High (4)	High	121.385	1.204	121.745	1.135	0.360	0.261	120.892	0.773	121.829	1.131	0.937	0.429	120.473	0.490	122.083	1.123	1.610	0.682
		Uniform	118.675	1.093	119.766	1.039	1.091	0.279	119.115	0.707	119.365	1.042	0.250	0.433	119.230	0.451	119.529	1.032	0.299	0.642
		Medium	115.469	1.109	115.552	1.049	0.083	0.262	116.224	0.721	115.610	1.051	-0.614	0.411	116.784	0.462	115.492	1.040	-1.292	0.631
		Low	116.140	0.997	116.479	0.989	0.339	0.298	116.065	0.653	116.224	0.994	0.159	0.442	115.773	0.419	116.527	0.989	0.754	0.636
Upward, Fast	Low (1)	High	108.641	1.180	108.434	1.176	-0.207	0.220	108.050	0.763	108.372	1.173	0.322	0.470	107.664	0.483	108.253	1.167	0.589	0.726
		Uniform	105.868	1.183	105.588	1.179	-0.280	0.248	105.969	0.763	105.568	1.177	-0.401	0.489	106.101	0.484	105.380	1.161	-0.721	0.727
		Medium	105.123	1.178	105.289	1.182	0.166	0.212	104.935	0.769	105.202	1.182	0.267	0.471	104.824	0.489	105.018	1.176	0.194	0.728
		Low	102.972	1.178	102.898	1.202	-0.074	0.244	103.030	0.764	102.859	1.201	-0.171	0.500	102.935	0.483	103.065	1.189	0.130	0.750
	Med. (1.85)	High	112.515	1.219	112.482	1.174	-0.034	0.231	112.838	0.788	112.535	1.172	-0.304	0.447	112.922	0.503	112.621	1.158	-0.301	0.698
		Uniform	112.463	1.252	112.529	1.215	0.066	0.251	112.665	0.810	112.504	1.213	-0.161	0.472	112.760	0.510	112.563	1.203	-0.197	0.740
		Medium	107.871	1.222	107.977	1.191	0.106	0.226	109.153	0.788	108.068	1.189	-1.086	0.459	109.897	0.497	108.296	1.177	-1.601	0.721
		Low	112.912	1.223	112.389	1.187	-0.524	0.265	112.157	0.796	112.357	1.186	0.200	0.468	111.701	0.506	112.217	1.175	0.517	0.719
	High (4)	High	120.389	1.276	120.255	1.188	-0.133	0.244	120.020	0.817	120.217	1.187	0.197	0.431	119.749	0.515	120.151	1.175	0.403	0.701
		Uniform	119.496	1.267	119.767	1.190	0.271	0.264	119.775	0.821	119.680	1.187	-0.095	0.446	120.033	0.522	119.519	1.172	-0.515	0.702
		Medium	120.662	1.204	120.554	1.146	-0.109	0.237	119.801	0.778	120.487	1.141	0.686	0.428	119.232	0.492	120.521	1.135	1.289	0.688
		Low	119.121	1.229	118.618	1.142	-0.503	0.268	119.116	0.797	118.545	1.139	-0.571	0.427	119.089	0.506	118.590	1.127	-0.499	0.677
Downward, Slow	Low (1)	High	67.261	1.017	86.524	1.005	19.264	0.496	71.209	0.812	83.976	1.026	12.767	0.525	74.171	0.657	81.558	1.085	7.387	0.620
		Uniform	50.471	0.923	68.467	0.944	17.995	0.422	54.260	0.784	60.598	1.128	6.339	0.495	56.349	0.656	58.404	1.186	2.055	0.633
		Medium	51.630	0.790	64.526	0.834	12.897	0.316	54.599	0.627	61.310	0.887	6.710	0.383	56.288	0.488	59.496	0.926	3.208	0.527
		Low	32.152	0.745	37.312	1.012	5.160	0.390	33.716	0.627	36.142	0.991	2.426	0.427	34.778	0.517	35.409	1.004	0.631	0.533
	Med. (1.85)	High	76.248	1.013	97.717	0.924	21.469	0.567	81.137	0.816	96.254	0.951	15.117	0.576	85.044	0.670	94.409	0.982	9.365	0.610
		Uniform	59.768	1.016	82.616	0.960	22.848	0.472	64.266	0.874	76.598	1.085	12.331	0.459	67.419	0.748	72.654	1.202	5.235	0.589
		Medium	60.060	0.815	74.787	0.826	14.727	0.327	63.761	0.655	72.905	0.835	9.143	0.365	66.395	0.524	71.311	0.860	4.916	0.471
		Low	41.570	0.832	52.487	1.023	10.917	0.359	43.391	0.719	47.489	1.105	4.098	0.469	44.480	0.606	46.523	1.100	2.043	0.553
	High (4)	High	91.619	1.028	111.138	0.907	19.519	0.525	95.982	0.815	110.140	0.923	14.159	0.534	99.492	0.662	108.666	0.947	9.174	0.569
		Uniform	74.656	1.016	100.904	0.844	26.248	0.545	79.003	0.874	97.260	0.900	18.256	0.476	82.548	0.758	94.012	0.961	11.464	0.486
		Medium	74.268	0.828	89.680	0.768	15.413	0.370	77.776	0.687	88.112	0.787	10.336	0.376	80.415	0.568	87.282	0.802	6.867	0.427
		Low	53.707	0.925	73.947	0.930	20.240	0.359	56.166	0.817	63.759	1.112	7.593	0.402	57.587	0.701	60.587	1.192	3.000	0.555
Downward, Fast	Low (1)	High	31.570	0.180	39.407	0.428	7.836	0.392	32.741	0.075	36.098	0.397	3.357	0.349	33.443	0.042	35.182	0.371	1.740	0.341
		Uniform	27.885	0.181	37.778	0.373	9.893	0.357	28.769	0.077	30.035	0.329	1.266	0.270	28.872	0.037	29.010	0.368	0.138	0.344
		Medium	24.556	0.120	26.903	0.264	2.347	0.209	25.165	0.061	26.051	0.242	0.886	0.204	25.554	0.037	25.719	0.232	0.165	0.211
		Low	21.277	0.140	21.630	0.275	0.353	0.158	21.294	0.064	21.490	0.268	0.196	0.223	21.280	0.036	21.456	0.275	0.176	0.255
	Med. (1.85)	High	39.333	0.199	53.335	0.530	14.001	0.483	41.073	0.102	47.159	0.471	6.086	0.398	42.113	0.065	45.174	0.440	3.061	0.389
		Uniform	35.297	0.202	50.880	0.483	15.584	0.452	37.096	0.102	43.185	0.368	6.089	0.309	37.956	0.044	39.654	0.349	1.698	0.321
		Medium	30.868	0.152	34.562	0.342	3.694	0.251	31.495	0.082	33.849	0.314	2.354	0.255	31.884	0.049	32.953	0.287	1.069	0.257
		Low	27.310	0.166	32.876	0.294	5.566	0.196	27.782	0.077	27.852	0.308	0.071	0.250	27.906	0.037	27.641	0.315	-0.266	0.292
	High (4)	High	51.002	0.260	72.960	0.593	21.958	0.493	53.643	0.174	64.189	0.552	10.546	0.417	55.458	0.133	60.891	0.536	5.433	0.426
		Uniform	47.249	0.213	72.830	0.605	25.581	0.555	50.086	0.148	64.631	0.545	14.545	0.461	51.963	0.096	57.936	0.419	5.973	0.347
		Medium	40.223	0.192	47.870	0.411	7.648	0.325	41.270	0.126	44.444	0.374	3.174	0.278	42.039	0.087	42.955	0.331	0.917	0.270
		Low	37.973	0.214	55.770	0.421	17.797	0.372	39.503	0.128	43.710	0.356	4.207	0.261	39.955	0.067	40.478	0.371	0.523	0.321
		Average	91.477	0.821	96.184	0.868	4.707	0.350	92.429	0.554	94.752	0.869	2.323	0.426	93.017	0.374	94.153	0.866	1.137	0.556

Note: Demand and cost samples are matched within each row, but independent across rows.

Table 7: Detailed results for the inventory example with the perfect information relaxation

Imperfect Information Relaxation

Transition Matrix	Shortage Penalty	Prior	Myopic Lower	MSE	Myopic Upper	MSE	Myopic Gap	MSE	One-period look-ahead Lower	MSE	One-period look-ahead Upper	MSE	One-period look-ahead Gap	MSE	Two-period look-ahead Lower	MSE	Two-period look-ahead Upper	MSE	Two-period look-ahead Gap	MSE
Stable, Positive Correlation	Low (1)	High	92.854	0.408	99.521	0.509	6.667	0.248	95.511	0.295	99.109	0.503	3.598	0.273	96.979	0.206	98.829	0.499	1.850	0.321
		Uniform	86.832	0.445	93.985	0.566	7.152	0.231	89.066	0.331	92.033	0.612	2.967	0.315	90.043	0.240	91.660	0.627	1.617	0.403
		Medium	85.113	0.371	89.662	0.510	4.549	0.227	87.198	0.268	89.277	0.507	2.079	0.280	88.304	0.185	88.736	0.508	0.431	0.341
		Low	76.641	0.480	80.088	0.713	3.446	0.278	78.214	0.351	79.891	0.707	1.677	0.375	78.975	0.250	79.501	0.702	0.525	0.463
	Med. (1.85)	High	106.010	0.273	110.357	0.431	4.347	0.222	107.906	0.203	110.206	0.428	2.301	0.256	108.901	0.147	110.044	0.414	1.143	0.286
		Uniform	101.137	0.351	106.126	0.505	4.989	0.217	103.054	0.266	105.354	0.519	2.299	0.279	104.004	0.196	105.015	0.525	1.011	0.343
		Medium	98.764	0.276	101.734	0.451	2.970	0.220	100.119	0.201	101.499	0.446	1.380	0.267	100.807	0.142	101.375	0.435	0.568	0.306
		Low	92.842	0.409	96.062	0.635	3.220	0.253	93.957	0.302	95.150	0.647	1.193	0.357	94.385	0.215	95.081	0.632	0.697	0.424
	High (4)	High	120.886	0.126	122.978	0.331	2.092	0.227	121.731	0.100	122.831	0.323	1.100	0.234	122.139	0.076	122.765	0.311	0.626	0.242
		Uniform	119.388	0.148	121.614	0.371	2.227	0.241	120.265	0.117	121.283	0.368	1.018	0.261	120.598	0.088	121.115	0.358	0.517	0.276
		Medium	115.767	0.130	117.449	0.350	1.682	0.238	116.319	0.096	117.256	0.342	0.938	0.255	116.545	0.069	117.342	0.326	0.797	0.262
		Low	114.695	0.171	116.050	0.420	1.355	0.263	115.282	0.131	115.757	0.414	0.475	0.291	115.508	0.095	115.804	0.397	0.296	0.307
Stable, Negative Correlation	Low (1)	High	99.154	0.038	99.646	0.284	0.492	0.317	99.212	0.007	99.472	0.268	0.261	0.262	99.206	0.001	99.386	0.252	0.180	0.253
		Uniform	98.759	0.038	98.844	0.297	0.085	0.330	98.761	0.007	98.735	0.287	-0.026	0.281	98.764	0.001	98.756	0.265	-0.008	0.266
		Medium	97.418	0.035	97.833	0.287	0.415	0.317	97.480	0.007	97.734	0.270	0.254	0.265	97.473	0.001	97.673	0.254	0.200	0.255
		Low	95.819	0.036	95.988	0.294	0.169	0.324	95.829	0.007	96.028	0.284	0.199	0.278	95.825	0.001	95.984	0.271	0.159	0.272
	Med. (1.85)	High	113.363	0.031	113.553	0.300	0.190	0.319	113.349	0.006	113.502	0.278	0.152	0.275	113.353	0.001	113.648	0.261	0.295	0.262
		Uniform	113.368	0.031	113.343	0.311	-0.025	0.331	113.362	0.006	113.314	0.299	-0.048	0.295	113.365	0.001	113.343	0.283	-0.022	0.284
		Medium	111.743	0.031	111.675	0.310	-0.068	0.330	111.735	0.006	111.633	0.295	-0.102	0.292	111.738	0.001	111.794	0.270	0.056	0.270
		Low	111.231	0.031	111.677	0.313	0.446	0.331	111.240	0.006	111.717	0.292	0.477	0.289	111.241	0.001	111.690	0.275	0.449	0.275
	High (4)	High	127.844	0.022	127.806	0.326	-0.038	0.330	127.891	0.004	127.942	0.307	0.050	0.306	127.885	0.001	128.013	0.293	0.128	0.293
		Uniform	128.330	0.022	128.480	0.334	0.151	0.337	128.342	0.004	128.472	0.320	0.130	0.320	128.336	0.001	128.511	0.301	0.175	0.301
		Medium	126.593	0.021	126.812	0.320	0.220	0.324	126.577	0.004	126.828	0.302	0.251	0.301	126.579	0.001	126.770	0.283	0.190	0.283
		Low	126.826	0.022	126.633	0.321	-0.193	0.323	126.854	0.004	126.679	0.303	-0.175	0.303	126.849	0.001	126.812	0.278	-0.037	0.278
Stable, Zero Correlation	Low (1)	High	100.528	0.004	100.732	0.323	0.203	0.324	100.557	0.001	100.644	0.302	0.087	0.302	100.557	0.000	100.611	0.285	0.054	0.285
		Uniform	99.782	0.001	99.562	0.330	-0.220	0.331	99.801	0.000	99.682	0.315	-0.119	0.315	99.801	0.000	99.694	0.295	-0.107	0.295
		Medium	98.181	0.001	98.251	0.335	0.071	0.335	98.197	0.000	98.211	0.315	0.014	0.315	98.198	0.000	98.424	0.295	0.226	0.295
		Low	96.610	0.001	97.003	0.329	0.393	0.329	96.627	0.000	96.806	0.305	0.179	0.305	96.628	0.000	96.902	0.292	0.274	0.292
	Med. (1.85)	High	114.426	0.003	114.560	0.336	0.135	0.337	114.437	0.001	114.661	0.316	0.224	0.316	114.437	0.000	114.608	0.293	0.171	0.293
		Uniform	114.211	0.000	114.578	0.339	0.367	0.339	114.212	0.000	114.558	0.320	0.346	0.320	114.212	0.000	114.555	0.301	0.343	0.301
		Medium	112.242	0.000	112.110	0.321	-0.133	0.321	112.243	0.000	112.086	0.302	-0.157	0.302	112.243	0.000	112.291	0.280	0.048	0.280
		Low	111.945	0.000	112.804	0.343	0.859	0.344	111.946	0.000	112.509	0.326	0.563	0.326	111.946	0.000	112.407	0.302	0.461	0.302
	High (4)	High	128.734	0.002	128.499	0.342	-0.235	0.342	128.745	0.000	128.594	0.318	-0.151	0.318	128.745	0.000	128.721	0.292	-0.024	0.292
		Uniform	129.094	0.000	129.147	0.328	0.053	0.328	129.095	0.000	129.024	0.311	-0.071	0.311	129.095	0.000	128.911	0.292	-0.184	0.292
		Medium	127.064	0.000	126.971	0.326	-0.093	0.326	127.063	0.000	127.223	0.310	0.160	0.310	127.063	0.000	127.222	0.288	0.158	0.288
		Low	127.586	0.000	127.381	0.324	-0.204	0.324	127.586	0.000	127.444	0.305	-0.142	0.305	127.586	0.000	127.493	0.288	-0.093	0.288
Upward, Slow	Low (1)	High	103.486	0.266	104.633	0.321	1.148	0.191	103.921	0.200	104.594	0.310	0.672	0.192	104.183	0.150	104.506	0.318	0.323	0.210
		Uniform	95.868	0.472	98.708	0.516	2.839	0.184	96.440	0.368	97.283	0.579	0.843	0.253	96.637	0.279	96.779	0.617	0.142	0.358
		Medium	97.524	0.251	98.363	0.358	0.839	0.185	97.748	0.190	98.451	0.354	0.703	0.218	97.842	0.145	98.204	0.362	0.362	0.249
		Low	85.142	0.546	85.246	0.726	0.104	0.221	85.107	0.412	85.139	0.728	0.032	0.340	85.089	0.305	85.255	0.712	0.166	0.423
	Med. (1.85)	High	111.106	0.176	112.077	0.256	0.971	0.180	111.459	0.131	112.017	0.251	0.558	0.182	111.688	0.099	112.070	0.241	0.382	0.186
		Uniform	106.414	0.328	108.607	0.405	2.193	0.177	107.013	0.253	108.083	0.429	1.070	0.221	107.309	0.192	107.857	0.442	0.548	0.274
		Medium	106.457	0.199	106.987	0.325	0.529	0.187	106.599	0.149	106.958	0.321	0.359	0.214	106.692	0.112	106.919	0.315	0.227	0.232
		Low	97.033	0.425	98.398	0.582	1.365	0.208	97.246	0.323	97.429	0.605	0.183	0.308	97.279	0.241	97.277	0.607	-0.002	0.382
	High (4)	High	119.601	0.036	120.062	0.189	0.461	0.177	119.787	0.025	120.115	0.178	0.328	0.169	119.900	0.019	120.208	0.168	0.309	0.162
		Uniform	119.053	0.082	120.215	0.242	1.162	0.192	119.370	0.063	119.762	0.255	0.393	0.211	119.480	0.048	119.839	0.242	0.359	0.209
		Medium	117.256	0.053	117.363	0.223	0.106	0.191	117.292	0.039	117.352	0.215	0.060	0.195	117.306	0.030	117.244	0.207	-0.062	0.193
		Low	115.170	0.134	115.746	0.321	0.576	0.215	115.237	0.100	115.425	0.323	0.188	0.242	115.239	0.074	115.493	0.317	0.254	0.256
Upward, Fast	Low (1)	High	107.093	0.004	106.977	0.176	-0.116	0.174	107.097	0.001	106.922	0.170	-0.175	0.169	107.097	0.000	106.856	0.163	-0.241	0.163
		Uniform	106.233	0.010	106.155	0.202	-0.079	0.196	106.233	0.002	106.119	0.197	-0.114	0.196	106.232	0.000	106.091	0.192	-0.141	0.192
		Medium	104.766	0.005	104.857	0.174	0.091	0.172	104.762	0.001	104.808	0.168	0.046	0.167	104.761	0.000	104.716	0.160	-0.045	0.160
		Low	102.902	0.012	102.818	0.203	-0.083	0.195	102.901	0.002	102.773	0.197	-0.128	0.195	102.901	0.000	102.769	0.192	-0.132	0.191
	Med. (1.85)	High	113.261	0.004	113.318	0.176	0.057	0.178	113.266	0.001	113.336	0.169	0.070	0.170	113.267	0.000	113.254	0.162	-0.012	0.162
		Uniform	113.146	0.004	113.296	0.197	0.150	0.200	113.142	0.002	113.225	0.189	0.082	0.191	113.141	0.001	113.220	0.180	0.079	0.181
		Medium	111.167	0.003	111.185	0.176	0.018	0.178	111.166	0.001	111.265	0.170	0.099	0.171	111.166	0.000	111.317	0.162	0.151	0.162
		Low	110.955	0.005	110.631	0.202	-0.324	0.206	110.948	0.002	110.599	0.194	-0.350	0.196	110.945	0.001	110.554	0.182	-0.391	0.183
	High (4)	High	119.626	0.010	119.484	0.171	-0.142	0.175	119.650	0.003	119.484	0.162	-0.166	0.164	119.656	0.001	119.455	0.155	-0.201	0.155
		Uniform	120.296	0.017	120.515	0.180	0.219	0.190	120.305	0.005	120.463	0.172	0.158	0.175	120.309	0.002	120.456	0.163	0.148	0.164
		Medium	118.153	0.011	118.146	0.171	-0.007	0.176	118.147	0.004	118.098	0.162	-0.049	0.164	118.145	0.001	118.129	0.156	-0.016	0.156
		Low	119.137	0.020	118.625	0.178	-0.512	0.191	119.117	0.006	118.576	0.174	-0.541	0.178	119.110	0.002	118.535	0.166	-0.575	0.167
Downward, Slow	Low (1)	High	66.573	0.828	85.541	0.767	18.968	0.474	70.835	0.736	83.176	0.805	12.341	0.420	73.976	0.627	80.737	0.862	6.761	0.401
		Uniform	51.282	0.849	69.388	0.855	18.106	0.391	54.692	0.766	61.516	1.049	6.824	0.417	56.558	0.655	59.339	1.103	2.780	0.505
		Medium	51.720	0.668	64.572	0.680	12.851	0.282	54.664	0.583	61.324	0.733	6.660	0.263	56.339	0.474	59.565	0.774	3.226	0.351
		Low	32.116	0.678	37.081	0.939	4.965	0.367	33.676	0.599	35.859	0.910	2.183	0.368	34.745	0.505	34.999	0.906	0.254	0.424
	Med. (1.85)	High	76.613	0.819	98.000	0.684	21.387	0.539	81.385	0.741	96.453	0.714	15.068	0.478	85.169	0.643	94.786	0.754	9.618	0.405
		Uniform	59.991	0.894	82.579	0.805	22.588	0.443	64.319	0.826	76.684	0.943	12.366	0.361	67.393	0.728	72.572	1.063	5.179	0.427
		Medium	60.584	0.679	75.205	0.657	14.621	0.291	64.025	0.605	73.319	0.679	9.294	0.260	66.546	0.509	71.641	0.709	5.094	0.293
		Low	41.852	0.785	52.786	0.969	10.934	0.340	43.514	0.704	47.765	1.047	4.250	0.420	44.531	0.602	46.806	1.040	2.275	0.477
	High (4)	High	91.711	0.774	111.133	0.631	19.422	0.489	95.984	0.709	110.106	0.650	14.122	0.445	99.485	0.620	108.643	0.685	9.157	0.363
		Uniform	74.512	0.874	100.693	0.668	26.181	0.521	78.913	0.818	97.031	0.736	18.118	0.414	82.497	0.736	93.805	0.809	11.308	0.342
		Medium	74.520	0.700	89.870	0.636	15.350	0.336	77.899	0.638	88.388	0.655	10.489	0.299	80.486	0.550	87.545	0.671	7.059	0.287
		Low	53.231	0.838	73.480	0.836	20.249	0.330	55.906	0.783	63.281	1.025	7.376	0.336	57.444	0.687	60.103	1.109	2.659	0.462
Downward, Fast	Low (1)	High	31.514	0.126	39.524	0.394	8.009	0.376	32.717	0.047	36.051	0.350	3.334	0.317	33.423	0.022	35.110	0.314	1.687	0.295
		Uniform	27.650	0.139	37.550	0.342	9.900	0.343	28.668	0.052	29.730	0.256	1.062	0.209	28.806	0.014	28.682	0.290	-0.124	0.280
		Medium	24.551	0.070	26.951	0.230	2.400	0.192	25.129	0.025	26.088	0.197	0.959	0.176	25.524	0.015	25.753	0.179	0.229	0.171
		Low	21.101	0.101	21.427	0.228	0.325	0.134	21.199	0.031	21.235	0.215	0.036	0.186	21.216	0.008	21.193	0.218	-0.022	0.210
	Med. (1.85)	High	39.244	0.146	53.241	0.505	13.997	0.467	41.028	0.084	47.034	0.436	6.006	0.370	42.086	0.056	45.010	0.395	2.924	0.347
		Uniform	35.300	0.170	51.015	0.471	15.715	0.443	37.059	0.086	43.347	0.340	6.288	0.283	37.921	0.029	39.816	0.312	1.895	0.287
		Medium	30.812	0.119	34.593	0.328	3.781	0.241	31.458	0.061	33.819	0.296	2.361	0.241	31.863	0.035	32.904	0.265	1.041	0.241
		Low	27.388	0.137	32.868	0.262	5.480	0.175	27.827	0.056	27.881	0.263	0.054	0.211	27.929	0.017	27.702	0.267	-0.227	0.251
	High (4)	High	51.191	0.191	73.039	0.572	21.849	0.463	53.722	0.156	64.208	0.522	10.485	0.386	55.481	0.126	60.938	0.501	5.456	0.387
		Uniform	47.296	0.171	72.979	0.595	25.683	0.532	50.103	0.133	64.764	0.539	14.661	0.451	51.963	0.086	58.066	0.412	6.102	0.337
		Medium	40.076	0.140	47.784	0.399	7.708	0.293	41.192	0.105	44.277	0.355	3.085	0.259	41.995	0.075	42.790	0.308	0.795	0.251
		Low	37.864	0.177	55.748	0.410	17.884	0.350	39.437	0.110	43.588	0.337	4.151	0.241	39.919	0.053	40.367	0.345	0.448	0.299
		Average	91.563	0.216	96.297	0.405	4.734	0.287	92.489	0.172	94.861	0.402	2.372	0.278	93.059	0.137	94.254	0.398	1.195	0.286

Note: Demand samples are matched within each row, but independent across rows.
Ordering costs are are evaluated explicitly (not through simulation) for both upper and lower bounds.

Table 8: Detailed results for the inventory example with the imperfect information relaxation

Modified Myopic Bounds / Best Myopic Policies

Transition	Shortage		Perfect Information Relaxation						Imperfect Information Relaxation						Best Myopic Policies (based on imperfect information bounds)					
Matrix	Penalty	Prior	Lower	MSE	Upper	MSE	Gap	MSE	Lower	MSE	Upper	MSE	Gap	MSE	Lower	Method	Upper	Method	Gap	% Gap
Stable, Positive Correlation	Low (1)	High	97.927	0.204	98.869	1.034	0.941	0.945	97.937	0.204	98.678	0.579	0.742	0.408	97.937	Modified	98.678	Modified	0.742	0.8%
		Uniform	90.054	0.173	91.318	1.054	1.265	0.967	90.055	0.173	91.566	0.685	1.511	0.543	90.055	Modified	91.566	Modified	1.511	1.7%
		Medium	88.654	0.156	88.250	0.980	-0.404	0.909	88.653	0.155	89.002	0.564	0.349	0.429	88.653	Modified	89.002	Modified	0.349	0.4%
		Low	79.019	0.139	78.105	1.025	-0.914	0.962	79.007	0.137	79.543	0.732	0.537	0.630	79.007	Modified	79.543	Modified	0.537	0.7%
	Med. (1.85)	High	109.168	0.224	109.552	0.973	0.383	0.884	109.195	0.222	110.185	0.471	0.989	0.316	109.195	Modified	110.185	Modified	0.989	0.9%
		Uniform	103.161	0.206	104.221	1.000	1.060	0.900	103.206	0.205	105.471	0.608	2.265	0.454	103.206	Modified	105.471	Modified	2.265	2.2%
		Medium	100.460	0.182	101.267	0.943	0.807	0.849	100.470	0.179	101.626	0.485	1.156	0.343	100.470	Modified	101.626	Modified	1.156	1.1%
		Low	93.784	0.150	95.342	1.058	1.558	1.008	93.801	0.147	95.255	0.674	1.454	0.623	93.801	Modified	95.255	Modified	1.454	1.5%
	High (4)	High	121.434	0.258	124.677	0.892	3.243	0.821	121.332	0.255	123.007	0.349	1.675	0.249	121.332	Modified	122.978	Original	1.646	1.3%
		Uniform	118.909	0.232	121.840	0.921	2.931	0.833	118.953	0.229	122.778	0.410	3.825	0.276	119.388	Original	121.614	Original	2.226	1.8%
		Medium	115.177	0.203	117.061	0.912	1.883	0.830	115.217	0.198	117.928	0.365	2.711	0.246	115.767	Original	117.449	Original	1.682	1.4%
		Low	111.874	0.183	118.923	0.881	7.049	0.839	111.837	0.180	118.322	0.434	6.485	0.407	114.695	Original	116.050	Original	1.355	1.2%
Stable, Negative Correlation	Low (1)	High	98.317	0.099	99.909	0.774	1.592	0.753	98.349	0.097	100.199	0.284	1.850	0.209	99.154	Original	99.646	Original	0.492	0.5%
		Uniform	97.760	0.102	99.359	0.797	1.599	0.772	97.813	0.100	99.328	0.297	1.515	0.218	98.759	Original	98.844	Original	0.085	0.1%
		Medium	96.512	0.098	97.875	0.790	1.363	0.770	96.551	0.095	98.365	0.286	1.814	0.213	97.418	Original	97.833	Original	0.415	0.4%
		Low	94.891	0.094	96.058	0.802	1.167	0.784	94.949	0.093	96.468	0.293	1.520	0.221	95.819	Original	95.988	Original	0.169	0.2%
	Med. (1.85)	High	112.557	0.154	114.714	0.779	2.157	0.691	112.575	0.143	114.455	0.298	1.880	0.178	113.363	Original	113.553	Original	0.190	0.2%
		Uniform	112.276	0.153	114.443	0.774	2.167	0.692	112.318	0.146	114.293	0.309	1.975	0.184	113.368	Original	113.343	Original	-0.025	0.0%
		Medium	110.734	0.153	113.096	0.796	2.362	0.710	110.732	0.144	112.482	0.308	1.749	0.188	111.743	Original	111.675	Original	-0.068	-0.1%
		Low	110.425	0.154	112.865	0.842	2.440	0.753	110.448	0.143	112.661	0.311	2.213	0.190	111.231	Original	111.677	Original	0.446	0.4%
	High (4)	High	126.674	0.209	127.573	0.765	0.899	0.632	126.886	0.193	129.061	0.325	2.176	0.151	127.844	Original	127.806	Original	-0.038	0.0%
		Uniform	127.342	0.210	128.820	0.771	1.478	0.641	127.478	0.196	129.788	0.332	2.310	0.155	128.330	Original	128.480	Original	0.150	0.1%
		Medium	125.888	0.199	128.283	0.764	2.395	0.637	125.909	0.184	128.168	0.319	2.259	0.152	126.593	Original	126.812	Original	0.219	0.2%
		Low	125.878	0.205	128.357	0.784	2.479	0.651	125.855	0.187	127.875	0.320	2.020	0.152	126.826	Original	126.633	Original	-0.193	-0.2%
Stable, Zero Correlation	Low (1)	High	99.023	0.122	101.664	0.842	2.641	0.814	99.048	0.121	101.702	0.326	2.653	0.228	100.528	Original	100.732	Original	0.204	0.2%
		Uniform	98.090	0.122	101.580	0.845	3.490	0.814	98.139	0.120	100.569	0.332	2.431	0.233	99.782	Original	99.562	Original	-0.220	-0.2%
		Medium	96.716	0.119	98.891	0.815	2.175	0.782	96.734	0.117	99.220	0.336	2.486	0.242	98.181	Original	98.251	Original	0.070	0.1%
		Low	95.282	0.113	99.007	0.858	3.725	0.828	95.317	0.111	97.921	0.330	2.604	0.245	96.610	Original	97.003	Original	0.393	0.4%
	Med. (1.85)	High	112.915	0.183	114.915	0.810	2.001	0.716	113.001	0.177	115.662	0.338	2.661	0.189	114.426	Original	114.560	Original	0.134	0.1%
		Uniform	112.735	0.183	115.009	0.841	2.273	0.742	112.807	0.174	115.706	0.340	2.900	0.194	114.211	Original	114.578	Original	0.367	0.3%
		Medium	110.724	0.176	113.614	0.804	2.890	0.707	110.709	0.166	113.112	0.322	2.403	0.185	112.242	Original	112.110	Original	-0.132	-0.1%
		Low	110.541	0.175	113.073	0.831	2.531	0.740	110.612	0.168	114.115	0.344	3.504	0.207	111.945	Original	112.804	Original	0.859	0.8%
	High (4)	High	127.389	0.240	130.024	0.809	2.634	0.674	127.380	0.228	129.853	0.343	2.473	0.149	128.734	Original	128.499	Original	-0.235	-0.2%
		Uniform	127.811	0.236	129.938	0.802	2.127	0.668	127.902	0.221	130.550	0.329	2.648	0.143	129.094	Original	129.147	Original	0.053	0.0%
		Medium	125.826	0.230	128.912	0.816	3.086	0.683	125.812	0.216	128.411	0.327	2.600	0.144	127.064	Original	126.971	Original	-0.093	-0.1%
		Low	126.071	0.216	128.588	0.772	2.516	0.652	126.105	0.205	129.015	0.325	2.910	0.155	127.586	Original	127.381	Original	-0.205	-0.2%
Upward, Slow	Low (1)	High	104.276	0.067	102.815	1.170	-1.461	1.187	104.274	0.067	104.507	0.357	0.233	0.410	104.274	Modified	104.507	Modified	0.233	0.2%
		Uniform	96.317	0.115	97.310	1.258	0.993	1.316	96.324	0.116	96.848	0.649	0.524	0.755	96.324	Modified	96.848	Modified	0.524	0.5%
		Medium	97.875	0.064	97.604	1.127	-0.271	1.143	97.872	0.064	98.219	0.372	0.348	0.424	97.872	Modified	98.219	Modified	0.348	0.4%
		Low	84.688	0.140	85.751	1.248	1.063	1.332	84.697	0.138	85.262	0.730	0.565	0.862	85.142	Original	85.246	Original	0.104	0.1%
	Med. (1.85)	High	111.801	0.088	110.554	1.137	-1.247	1.150	111.803	0.087	111.949	0.272	0.145	0.342	111.803	Modified	111.949	Modified	0.145	0.1%
		Uniform	106.361	0.139	106.568	1.176	0.206	1.230	106.357	0.138	108.162	0.519	1.805	0.641	106.414	Original	108.162	Modified	1.748	1.6%
		Medium	106.460	0.089	106.777	1.091	0.316	1.109	106.464	0.089	107.019	0.340	0.555	0.410	106.464	Modified	106.987	Original	0.523	0.5%
		Low	96.750	0.243	97.803	1.141	1.052	1.281	96.755	0.242	97.578	0.618	0.823	0.848	97.033	Original	97.578	Modified	0.545	0.6%
	High (4)	High	119.679	0.116	121.665	1.142	1.986	1.154	119.653	0.114	119.977	0.196	0.324	0.286	119.653	Modified	119.977	Modified	0.324	0.3%
		Uniform	117.867	0.167	120.716	1.073	2.849	1.109	117.873	0.166	121.142	0.285	3.269	0.428	119.053	Original	120.215	Original	1.162	1.0%
		Medium	116.645	0.117	115.670	1.056	-0.975	1.073	116.661	0.117	117.490	0.227	0.830	0.322	117.256	Original	117.363	Original	0.107	0.1%
		Low	111.185	0.319	118.599	1.028	7.414	1.154	111.183	0.319	117.803	0.329	6.620	0.618	115.170	Original	115.746	Original	0.576	0.5%
Upward, Fast	Low (1)	High	107.096	0.012	108.434	1.176	1.338	1.177	107.098	0.012	106.978	0.176	-0.120	0.184	107.098	Modified	106.977	Original	-0.121	-0.1%
		Uniform	106.121	0.028	105.596	1.185	-0.525	1.189	106.125	0.028	106.167	0.202	0.041	0.221	106.233	Original	106.155	Original	-0.078	-0.1%
		Medium	104.740	0.014	105.290	1.183	0.550	1.183	104.740	0.014	104.857	0.174	0.117	0.182	104.766	Original	104.857	Original	0.091	0.1%
		Low	102.838	0.047	102.903	1.204	0.065	1.206	102.847	0.047	102.823	0.203	-0.024	0.235	102.902	Original	102.818	Original	-0.084	-0.1%
	Med. (1.85)	High	113.242	0.012	112.480	1.174	-0.762	1.175	113.243	0.013	113.319	0.176	0.076	0.183	113.261	Original	113.318	Original	0.057	0.1%
		Uniform	113.056	0.029	112.528	1.216	-0.528	1.218	113.064	0.029	113.297	0.197	0.232	0.216	113.146	Original	113.296	Original	0.150	0.1%
		Medium	111.138	0.020	107.977	1.192	-3.161	1.191	111.141	0.020	111.185	0.176	0.044	0.188	111.167	Original	111.185	Original	0.018	0.0%
		Low	110.936	0.050	112.548	1.198	1.611	1.203	110.935	0.050	110.780	0.202	-0.155	0.236	110.955	Original	110.631	Original	-0.324	-0.3%
	High (4)	High	119.642	0.012	120.255	1.189	0.613	1.189	119.642	0.012	119.484	0.171	-0.158	0.178	119.642	Modified	119.484	Original	-0.158	-0.1%
		Uniform	120.188	0.027	119.776	1.191	-0.412	1.194	120.193	0.027	120.529	0.180	0.335	0.196	120.296	Original	120.515	Original	0.219	0.2%
		Medium	118.138	0.022	120.558	1.147	2.420	1.147	118.135	0.022	118.148	0.171	0.013	0.182	118.153	Original	118.146	Original	-0.007	0.0%
		Low	119.097	0.047	118.731	1.149	-0.366	1.153	119.101	0.047	118.747	0.178	-0.353	0.207	119.137	Original	118.625	Original	-0.512	-0.4%
Downward, Slow	Low (1)	High	78.167	0.397	79.452	1.285	1.285	1.004	78.181	0.397	78.457	1.055	0.277	0.679	78.181	Modified	78.457	Modified	0.277	0.4%
		Uniform	58.467	0.398	57.162	1.235	-1.305	0.920	58.436	0.396	58.116	1.151	-0.320	0.764	58.436	Modified	58.116	Modified	-0.320	-0.5%
		Medium	57.803	0.304	58.689	1.046	0.887	0.821	57.780	0.303	58.747	0.893	0.967	0.599	57.780	Modified	58.747	Modified	0.967	1.7%
		Low	35.878	0.300	34.502	1.002	-1.376	0.745	35.897	0.301	34.340	0.916	-1.557	0.619	35.897	Modified	34.340	Modified	-1.557	-4.4%
	Med. (1.85)	High	91.753	0.502	92.064	1.168	0.310	0.852	91.787	0.502	92.348	0.953	0.561	0.529	91.787	Modified	92.348	Modified	0.561	0.6%
		Uniform	71.057	0.521	69.992	1.335	-1.064	0.913	71.064	0.521	70.094	1.199	-0.970	0.707	71.064	Modified	70.094	Modified	-0.970	-1.4%
		Medium	69.560	0.394	69.382	1.015	-0.179	0.724	69.574	0.394	69.853	0.865	0.280	0.491	69.574	Modified	69.853	Modified	0.280	0.4%
		Low	45.824	0.425	45.735	1.123	-0.089	0.752	45.827	0.425	46.109	1.064	0.282	0.652	45.827	Modified	46.109	Modified	0.282	0.6%
	High (4)	High	107.016	0.608	107.495	1.103	0.479	0.794	107.012	0.605	107.436	0.844	0.424	0.447	107.012	Modified	107.436	Modified	0.424	0.4%
		Uniform	88.396	0.639	88.048	1.276	-0.347	0.789	88.386	0.638	87.796	1.139	-0.590	0.581	88.386	Modified	87.796	Modified	-0.590	-0.7%
		Medium	84.463	0.511	84.949	0.957	0.486	0.607	84.483	0.512	85.181	0.838	0.698	0.396	84.483	Modified	85.181	Modified	0.698	0.8%
		Low	59.333	0.541	59.649	1.240	0.317	0.770	59.309	0.540	59.052	1.148	-0.257	0.643	59.309	Modified	59.052	Modified	-0.257	-0.4%
Downward, Fast	Low (1)	High	34.358	0.075	34.260	0.348	-0.099	0.318	34.369	0.073	34.171	0.268	-0.198	0.211	34.369	Modified	34.171	Modified	-0.198	-0.6%
		Uniform	28.812	0.073	29.039	0.367	0.227	0.320	28.824	0.073	28.686	0.292	-0.138	0.222	28.824	Modified	28.686	Modified	-0.138	-0.5%
		Medium	25.982	0.037	25.614	0.232	-0.368	0.218	25.970	0.035	25.723	0.174	-0.248	0.147	25.970	Modified	25.723	Modified	-0.248	-1.0%
		Low	21.204	0.046	21.460	0.277	0.256	0.247	21.217	0.046	21.167	0.219	-0.050	0.176	21.217	Modified	21.167	Modified	-0.050	-0.2%
	Med. (1.85)	High	43.975	0.139	44.324	0.401	0.348	0.356	43.979	0.139	44.094	0.342	0.115	0.285	43.979	Modified	44.094	Modified	0.115	0.3%
		Uniform	38.426	0.143	38.427	0.391	0.002	0.288	38.417	0.143	38.599	0.353	0.182	0.218	38.417	Modified	38.599	Modified	0.182	0.5%
		Medium	32.729	0.086	32.735	0.278	0.007	0.237	32.735	0.085	32.682	0.250	-0.053	0.193	32.735	Modified	32.682	Modified	-0.053	-0.2%
		Low	27.855	0.089	27.659	0.317	-0.197	0.251	27.856	0.089	27.684	0.269	-0.172	0.182	27.856	Modified	27.684	Modified	-0.172	-0.6%
	High (4)	High	58.461	0.285	58.261	0.487	-0.200	0.391	58.456	0.285	58.434	0.432	-0.022	0.327	58.456	Modified	58.434	Modified	-0.022	0.0%
		Uniform	53.852	0.253	53.851	0.389	0.000	0.257	53.851	0.253	53.911	0.375	0.060	0.209	53.851	Modified	53.911	Modified	0.060	0.1%
		Medium	43.087	0.138	42.475	0.312	-0.612	0.267	43.094	0.138	42.289	0.282	-0.806	0.224	43.094	Modified	42.289	Modified	-0.806	-1.9%
		Low	39.979	0.188	40.019	0.389	0.040	0.232	39.979	0.188	39.907	0.361	-0.072	0.179	39.979	Modified	39.907	Modified	-0.072	-0.2%
		Average	93.150	0.183	94.168	0.914	1.019	0.828	93.164	0.180	94.280	0.438	1.116	0.332	93.620		93.893		0.273	0.2%

Note: Demand samples are matched within each row, but independent across rows.
For the imperfect information bounds, ordering costs are evaluated explicitly (not through simulation) for both upper and lower bounds.

Table 9: Detailed results for the inventory example with the modified myopic policy