Online Appendix for

Technology Adoption with Uncertain Future Costs and Quality

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A. Proofs

A.1. Proof of Proposition 2.1: Comparing Values and Policies

The proof of the last part of Proposition 2.1 relies on the following lemma.

Lemma A.1. *Assume that the quality is almost certainly improving over time. For all k, c_k and $p_k \geq q_k^2 \geq q_k^1$, we have*

$$v_k^r(p_k, c_k, q_k^1) - v_k^r(p_k, c_k, q_k^2) + \frac{1-\delta^k}{1-\delta} q_k^2 \quad \leq \quad v_k^s(p_k, c_k, q_k^1).$$

Proof of Lemma A.1. The result is trivially true for $k = 0$. Assume inductively that this is true for $k-1$. Then

$$\begin{aligned}
& v_k^r(p_k, c_k, q_k^1) - v_k^r(p_k, c_k, q_k^2) + \frac{1-\delta^k}{1-\delta} q_k^2 \\
(1) \quad = \quad & \max\left\{ \begin{array}{l} p_k - c_k + \delta \mathbb{E}\big[v_{k-1}^r(\tilde{p}_{k-1}, \tilde{c}_{k-1}, p_k) \mid p_k, c_k\big], \\ q_k^1 + \delta \mathbb{E}\big[v_{k-1}^r(\tilde{p}_{k-1}, \tilde{c}_{k-1}, q_k^1) \mid p_k, c_k\big] \end{array} \right. \\
& - \max\left\{ \begin{array}{l} p_k - c_k + \delta \mathbb{E}\big[v_{k-1}^r(\tilde{p}_{k-1}, \tilde{c}_{k-1}, p_k) \mid p_k, c_k\big], \\ q_k^2 + \delta \mathbb{E}\big[v_{k-1}^r(\tilde{p}_{k-1}, \tilde{c}_{k-1}, q_k^2) \mid p_k, c_k\big] \end{array} \right. \quad + \quad \frac{1-\delta^k}{1-\delta} q_k^2 \\
(2) \quad = \quad & \max\left\{ \begin{array}{l} \frac{1-\delta^k}{1-\delta} p_k - c_k, \\ q_k^1 + \delta \mathbb{E}\Big[v_{k-1}^r(\tilde{p}_{k-1}, \tilde{c}_{k-1}, q_k^1) - v_{k-1}^r(\tilde{p}_{k-1}, \tilde{c}_{k-1}, p_k) + \frac{1-\delta^{k-1}}{1-\delta} p_k \mid p_k, c_k\Big] \end{array} \right. \\
& - \max\left\{ \begin{array}{l} \frac{1-\delta^k}{1-\delta} (p_k - q_k^2) - c_k, \\ \delta \mathbb{E}\Big[v_{k-1}^r(\tilde{p}_{k-1}, \tilde{c}_{k-1}, q_k^2) - v_{k-1}^r(\tilde{p}_{k-1}, \tilde{c}_{k-1}, p_k) + \frac{1-\delta^{k-1}}{1-\delta} (p_k - q_k^2) \mid p_k, c_k\Big] \end{array} \right. \\
(3) \quad \leq \quad & \max\left\{ \begin{array}{l} \frac{1-\delta^k}{1-\delta} p_k - c_k, \\ q_k^1 + \delta \mathbb{E}\Big[v_{k-1}^r(\tilde{p}_{k-1}, \tilde{c}_{k-1}, q_k^1) - v_{k-1}^r(\tilde{p}_{k-1}, \tilde{c}_{k-1}, p_k) + \frac{1-\delta^{k-1}}{1-\delta} p_k \mid p_k, c_k\Big] \end{array} \right. \\
(4) \quad \leq \quad & \max\left\{ \begin{array}{l} \frac{1-\delta^k}{1-\delta} p_k - c_k, \\ q_k^1 + \delta \mathbb{E}\big[v_{k-1}^s(\tilde{p}_{k-1}, \tilde{c}_{k-1}, q_k^1) \mid p_k, c_k\big] \end{array} \right. \\
(5) \quad = \quad & v_k^s(p_k, c_k, q_k^1)
\end{aligned}$$

Equality (1) uses the definition of v_k^r. Equality (2) follows from adding $\delta \frac{1-\delta^{k-1}}{1-\delta} p_k - \delta \mathbb{E}[v_k^r(\tilde{p}_{k-1}, \tilde{c}_{k-1}, p_k) | p_k, c_k]$ to both maxima in (1) (this cancels, since the maxima have opposite signs) and rearranging. Since $q_k^2 \leq p_k$, by Proposition 2.2(3), the second (subtracted) maximum in (2) is nonnegative; this term is dropped to give inequality (3). We then use the induction hypothesis to yield inequality (4). Equality (5) follows from the definition of v_k^s. □

We now present the proof of Proposition 2.1.

Proof of Proposition 2.1.

1. In the repeat-purchase model the consumer could always adopt once (as in the single-purchase model) but may also choose to adopt new versions of the technology later if this leads to a larger expected value. Similarly, in both the single- and repeat-purchase models, the consumer could adopt the technology now and hold it for all remaining periods if that is optimal.

2. Here we give a proof for the repeat-purchase model, assuming $c_k \geq 0$. We will show that the value from adoption is less than the value from waiting if the technology is not cost effective. Suppose the technology in the market is not cost effective and $p_k \leq q_k$. Because the optimal value function is increasing in the quality of the technology owned both the rewards (since $c_k \geq 0$) and the continuation value from waiting is higher than that of adoption.

   Now suppose that the technology in the market is not cost effective but $p_k \geq q_k$.

$$\text{value from adoption} = p_k - c_k + \delta \mathbb{E}\big[v_{k-1}^r(\tilde{p}_{k-1}, \tilde{c}_{k-1}, p_k) | p_k, c_k\big]$$

$$
\begin{aligned}
&= \frac{1-\delta^k}{1-\delta}p_k - c_k + \delta\mathbb{E}\Big[v^r_{k-1}(\tilde{p}_{k-1},\tilde{c}_{k-1},p_k) - \frac{1-\delta^{k-1}}{1-\delta}p_k|p_k,c_k\Big] \\
&\le \frac{1-\delta^k}{1-\delta}q_k + \delta\mathbb{E}\Big[v^r_{k-1}(\tilde{p}_{k-1},\tilde{c}_{k-1},p_k) - \frac{1-\delta^{k-1}}{1-\delta}p_k|p_k,c_k\Big] \\
&\le \frac{1-\delta^k}{1-\delta}q_k + \delta\mathbb{E}\Big[v^r_{k-1}(\tilde{p}_{k-1},\tilde{c}_{k-1},q_k) - \frac{1-\delta^{k-1}}{1-\delta}q_k|p_k,c_k\Big] \\
&= q_k + \delta\mathbb{E}\big[v^r_{k-1}(\tilde{p}_{k-1},\tilde{c}_{k-1},q_k)|p_k,c_k\big] \\
&= \text{value from waiting}
\end{aligned}
$$

The first inequality follows because the technology in the market is not cost effective; the second inequality follows because $p_k \ge q_k$ and $v_k(p_k,c_k,q_k) - \frac{1-\delta^k}{1-\delta}q_k$ is decreasing in q_k, as shown in Proposition 2.2(3).

3. $p_k - c_k \ge q_k$ implies that $p_k \ge q_k$ if $c_k \ge 0$. Then, the value from adoption is higher than the value from waiting because both the rewards and continuation values are higher (this follows because the optimal value function is increasing in the quality of the technology owned).

4. To show our final result, it is sufficient to show that the value from adoption minus the value from waiting is higher in the repeat-purchase model than in the single-purchase model. This implies that if it is optimal to adopt in the single-purchase model, then it is also optimal to adopt in the repeat-purchase model. We can write this as follows:

$$
\begin{aligned}
p_k - c_k - q_k + \delta\mathbb{E}\big[v^r_{k-1}(\tilde{p}_{k-1},\tilde{c}_{k-1},p_k) - v^r_{k-1}(\tilde{p}_{k-1},\tilde{c}_{k-1},q_k)|p_k,c_k\big] \\
\ge \frac{1-\delta^k}{1-\delta}p_k - c_k - q_k - \delta\mathbb{E}\big[v^s_{k-1}(\tilde{p}_{k-1},\tilde{c}_{k-1},q_k)|p_k,c_k\big]
\end{aligned}
$$

This is equivalent to proving

$$
\mathbb{E}\big[v^r_{k-1}(\tilde{p}_{k-1},\tilde{c}_{k-1},q_k) - v^r_{k-1}(\tilde{p}_{k-1},\tilde{c}_{k-1},p_k)|\ p_k,c_k\big] + \frac{1-\delta^{k-1}}{1-\delta}p_k \ \le \mathbb{E}\big[v^s_{k-1}(\tilde{p}_{k-1},\tilde{c}_{k-1},q_k)\ |\ p_k,c_k\big]\,.
$$

With the assumption that the quality p_k is almost certainly improving over time, this result follows from Lemma A.1. With non-decreasing qualities, we need only consider cases where $p_k \ge q_k$, since the consumer cannot own a technology better than the current one on the market.

□

A.2. Proof of Proposition 2.2: Impact of Changing the Quality of the Technology Owned

Proof. 1. This result holds trivially for $k = 0$. Assume that it holds when there are $k-1$ periods to go. The value from waiting is increasing in q_k because both the rewards and the continuation value is increasing (this follows from the induction hypothesis and because the transitions do not depend on q_k); the value from adoption does not depend on q_k, then, the optimal value functions are increasing in q_k because they are the maximum of increasing functions.

2. Proving the monotonicity of optimal policies is similar in all models. The value from adoption does not depend on q_k; any improvement in q_k will only increase the value from waiting. Then, if it is optimal to wait when the technology owned has quality q^1_k, it is also optimal to wait when the technology currently owned has quality $q^2_k \ge q^1_k$ keeping everything else the same.

3. Here we give the proof for the repeat-purchase model, the proofs for the other models are similar. The property holds trivially for $k = 0$. Assume that $v^r_{k-1}(p_{k-1},c_{k-1},q_{k-1}) - \frac{1-\delta^{k-1}}{1-\delta}q_{k-1}$ is decreasing in q_{k-1}. Then,

$$
v^r_k(p_k,c_k,q_k) - \frac{1-\delta^k}{1-\delta}q_k = \max\left\{\begin{array}{l} p_k - c_k - \frac{1-\delta^k}{1-\delta}q_k + \delta\mathbb{E}\big[v^r_{k-1}(\tilde{p}_{k-1},\tilde{c}_{k-1},p_k)|p_k,c_k\big]\,, \\ \delta\mathbb{E}\Big[v^r_{k-1}(\tilde{p}_{k-1},\tilde{c}_{k-1},q_k) - \frac{1-\delta^{k-1}}{1-\delta}q_k|p_k,c_k\Big] \end{array}\right.
$$

The first argument inside the maximization statement is decreasing because the rewards are decreasing in q_k and the continuation value does not depend on q_k. The second argument is also decreasing; this follows from the induction hypothesis and the fact that the transitions do not depend on q_k. Then, $v_k^r(p_k, c_k, q_k) - \frac{1-\delta^k}{1-\delta} q_k$ is decreasing in q_k, because it is the maximum of decreasing functions. $\square$

A.3. Proof of Proposition 3.3: Equivalent Conditions for CI-Dominance

Proof. See Theorem 1 in **?**. To relate our result, to the more general one in **?**, we note that the CI partial order is closed. In their result, equivalent condition (4) asserts the existence of $(\tilde{\pi}_k^2, \tilde{\varsigma}_k^2)$ and $(\tilde{\pi}_k^1, \tilde{\varsigma}_k^1)$ that are equal in distribution to $(\tilde{p}_k^2, \tilde{c}_k^2)$ and $(\tilde{p}_k^1, \tilde{c}_k^1)$ and such that $(\tilde{\pi}_k^2, \tilde{\varsigma}_k^2) \geq_{CI} (\tilde{\pi}_k^1, \tilde{\varsigma}_k^1)$ almost surely. Given the translation invariance of the CI-order, this is equivalent to condition (4) as stated in our proposition. $\square$

A.4. Proof that Additive Transitions Exhibit Diminishing Improvements

Proof. We show that the additive transitions of equation (1) exhibit diminishing improvements. Using equation (1) and some algebra, we have

$$\begin{aligned}
&\delta \mathbb{E}\Big[\frac{1-\delta^{k-1}}{1-\delta}\tilde{p}_{k-1} - \tilde{c}_{k-1} | p_k^2, c_k^2\Big] - \Big(\frac{1-\delta^k}{1-\delta} p_k^2 - c_k^2\Big) \\
&= \Big(\delta \frac{1-\delta^{k-1}}{1-\delta} - \frac{1-\delta^k}{1-\delta}\Big) p_k^2 + (1-\delta) c_k^2 + \delta \mathbb{E}\Big[\frac{1-\delta^{k-1}}{1-\delta} \tilde{u^p}_{k-1} - \tilde{u^c}_{k-1}\Big] \\
&= -(1-\delta)\big(\frac{1}{1-\delta} p_k^2 - c_k^2\big) + \delta \mathbb{E}\Big[\frac{1-\delta^{k-1}}{1-\delta} \tilde{u^p}_{k-1} - \tilde{u^c}_{k-1}\Big] \\
&\leq -(1-\delta)\big(\frac{1}{1-\delta} p_k^1 - c_k^1\big) + \delta \mathbb{E}\Big[\frac{1-\delta^{k-1}}{1-\delta} \tilde{u^p}_{k-1} - \tilde{u^c}_{k-1}\Big] \\
&= \delta \mathbb{E}\Big[\frac{1-\delta^{k-1}}{1-\delta}\tilde{p}_{k-1} - \tilde{c}_{k-1} | p_k^1, c_k^1\Big] - \Big(\frac{1-\delta^k}{1-\delta} p_k^1 - c_k^1\Big).
\end{aligned}$$

The inequality follows because $(\frac{1}{1-\delta} p_k - c_k)$ is CI-increasing $\square$

A.5. Proof of Simplified Representation with Additive Transitions.

Proof. We must show that equation (5) holds if transitions satisfy the additive model (1). The proof is by induction. For $k = 0$, the proof is trivial. For the induction hypothesis, assume $v_{k-1}^r(p_{k-1}, c_{k-1}, q_{k-1}) = \frac{1-\delta^{k-1}}{1-\delta} q_{k-1} + h_{k-1}^r(p_{k-1} - q_{k-1}, c_{k-1})$. Then,

$$\begin{aligned}
v_k^r(p_k, c_k, q_k) &= \max \left\{ \begin{array}{l} p_k - c_k + \delta \mathbb{E}\big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, p_k)\big], \\ q_k + \delta \mathbb{E}\big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, q_k)\big] \end{array} \right. \\
&= \frac{1-\delta^k}{1-\delta} q_k + \max \left\{ \begin{array}{l} \frac{1-\delta^k}{1-\delta}(p_k - q_k) - c_k + \delta \mathbb{E}\Big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, p_k) - \frac{1-\delta^{k-1}}{1-\delta} p_k\Big], \\ \delta \mathbb{E}\Big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, q_k) - \frac{1-\delta^{k-1}}{1-\delta} q_k\Big] \end{array} \right. \\
&= \frac{1-\delta^k}{1-\delta} q_k + \max \left\{ \begin{array}{l} \frac{1-\delta^k}{1-\delta}(p_k - q_k) - c_k + \delta \mathbb{E}\big[h_{k-1}^r(p_k + \tilde{u}_{k-1}^p - p_k, c_k + \tilde{u}_{k-1}^c)\big], \\ \delta \mathbb{E}\big[h_{k-1}^r(p_k - q_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c)\big] \end{array} \right. \\
&= \frac{1-\delta^k}{1-\delta} q_k + h_k^r(p_k - q_k, c_k)
\end{aligned}$$

where the third equality follows from the induction hypothesis. $\square$

A.6. Proof of Proposition 3.7: Independent Additive Increments: Repeat-Purchase Model

Proof of Proposition 3.7(1). Note that $h_k^r(\Delta_k, c_k) = v_k^r(\Delta_k, c_k, 0)$. Then, because additive transitions are CI-increasing, the result follows from Proposition 3.4. $\square$

Our proof of part (2) of Proposition 3.7 relies on the following lemma.

Lemma A.2. *If transitions satisfy the additive model* (1)*, then* $h_k^r(\Delta_k, c_k) - \frac{1-\delta^k}{1-\delta}\Delta_k$ *is decreasing in* Δ_k

Proof of Lemma A.2. The property holds trivially for $k = 0$. Assume that $h_{k-1}^r(\Delta_{k-1}, c_{k-1}) - \frac{1-\delta^{k-1}}{1-\delta}\Delta_{k-1}$ is decreasing in Δ_{k-1}. We then have:

$$
\begin{aligned}
&h_k^r(\Delta_k, c_k) - \frac{1-\delta^k}{1-\delta}\Delta_k \\
&\qquad = \max\left\{ \begin{array}{l} \frac{1-\delta^k}{1-\delta}\Delta_k - c_k + \delta\mathbb{E}\big[h_{k-1}^r(\tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c)\big] \\ \delta\mathbb{E}\big[h_{k-1}^r(\Delta_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c)\big] \end{array} \right. - \frac{1-\delta^k}{1-\delta}\Delta_k \\
&\qquad = \max\left\{ \begin{array}{l} -c_k + \delta\mathbb{E}\big[h_{k-1}^r(\tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c)\big] \\ -\Delta_k + \delta\frac{1-\delta^{k-1}}{1-\delta}\mathbb{E}\big[\tilde{u}_{k-1}^p\big] + \delta\mathbb{E}\Big[h_{k-1}^r(\Delta_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c) - \frac{1-\delta^{k-1}}{1-\delta}(\Delta_k + \tilde{u}_{k-1}^p)\Big] \end{array} \right. .
\end{aligned}
$$

The first expression inside the maximization statement does not depend on Δ_k. The second term in the maximization statement is decreasing in Δ_k: the reward term is decreasing in Δ_k and the expectation is decreasing in Δ_k by the induction hypothesis (the induction hypothesis holds for each realization of $\tilde{u}_{k-1}^p$ and $\tilde{u}_{k-1}^c$, then, it must also hold in expectation). Thus, both terms of the maximum are decreasing in Δ_k and $h_k^r(\Delta_k, c_k) - \frac{1-\delta^k}{1-\delta}\Delta_k$ is decreasing in Δ_k. □

Proof of Proposition 3.7(2). Let $g_k(\Delta_k, c_k)$ be the difference between the value from adopting and the value from waiting, i.e.,

$$
g_k(\Delta_k, c_k) = \frac{1-\delta^k}{1-\delta}\Delta_k - c_k + \delta\mathbb{E}\big[h_{k-1}^r(\tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c) - h_{k-1}^r(\Delta_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c)\big]\,. \tag{1}
$$

After rearranging, we have

$$
g_k(\Delta_k, c_k) = \Delta_k - c_k + \delta\mathbb{E}\left[h_{k-1}^r(\tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c) - \left(h_{k-1}^r(\Delta_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c) - \frac{1-\delta^{k-1}}{1-\delta}\Delta_k\right)\right].
$$

By Lemma A.2, $h_{k-1}^r(\Delta_k + u_{k-1}^p, c_k + u_{k-1}^c) - \frac{1-\delta^{k-1}}{1-\delta}\Delta_k$ is decreasing in Δ_k for each realization of u_{k-1}^p and u_{k-1}^c. Taking expectations over this, we find that $g_k(\Delta_k, c_k)$ is increasing in Δ_k.

If it is optimal to adopt with lag Δ_k^1, then $g_k(\Delta_k^1, c_k) \geq 0$. Because g_k is increasing in Δ_k, we have $g_k(\Delta_k^2, c_k) \geq g_k(\Delta_k^1, c_k) \geq 0$ for $\Delta_k^2 \geq \Delta_k^1$. This implies that it is also optimal to adopt for any lag $\Delta_k^2 \geq \Delta_k^1$. □

The proof of part (3) of Proposition 3.7 relies on the following lemma, in addition to Lemma 3.8 which is given in the body of the paper and proven after the proof of the proposition.

Lemma A.3. *Suppose that the quality of the technology is non-decreasing over time ($\tilde{u}_{k-1}^p \geq 0$ almost certainly) and that transitions satisfy the additive model* (1)*. Let $(\Delta_k^2, c_k^2) \geq_{CI} (\Delta_k^1, c_k^1)$ and $c_k^2 \leq c_k^1$ (that is, for changes in regions I or III of Figure 3). Then, for $u \geq 0$, we have*

$$
\frac{1-\delta^k}{1-\delta}\Delta_k^1 - c_k^1 + h_k^r(u, c_k^1) - h_k^r(\Delta_k^1 + u, c_k^1) \leq \frac{1-\delta^k}{1-\delta}\Delta_k^2 - c_k^2 + h_k^r(u, c_k^2) - h_k^r(\Delta_k^2 + u, c_k^2) \tag{2}
$$

Proof of Lemma A.3. The proof is by induction. The property holds trivially for $k = 0$. Assume that (2) holds for period k-1. Then,

$$
\frac{1-\delta^k}{1-\delta}\Delta_k - c_k + h_k^r(u, c_k) - h_k^r(\Delta_k + u, c_k) \tag{3}
$$

$$= \max \begin{cases} \frac{1-\delta^k}{1-\delta}(\Delta_k + u) - 2c_k & + \delta \mathbb{E}\big[h^r_{k-1}(\tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1})\big] \\ \frac{1-\delta^k}{1-\delta}\Delta_k - c_k & + \delta \mathbb{E}\big[h^r_{k-1}(u + \tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1})\big] \end{cases}$$

$$- \max \begin{cases} \frac{1-\delta^k}{1-\delta}(\Delta_k + u) - c_k & + \delta \mathbb{E}\big[h^r_{k-1}(\tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1})\big] \\ \delta \mathbb{E}\big[h^r_{k-1}(u + \tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1})\big] \end{cases}$$

Subtracting $\frac{1-\delta^k}{1-\delta}(\Delta_k + u) - c_k + \delta \mathbb{E}\big[h^r_{k-1}(\tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1})\big]$ from all terms in the maxima (the subtraction cancels in net), this is equal to:

$$= \max \begin{cases} -c_k \\ -\frac{1-\delta^k}{1-\delta}u + \delta \mathbb{E}\big[h^r_{k-1}(u + \tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1}) - h^r_{k-1}(\tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1})\big] \end{cases} \tag{4}$$

$$- \max \begin{cases} 0 \\ -\left(\frac{1-\delta^k}{1-\delta}(\Delta_k + u) - c_k\right) + \delta \mathbb{E}\big[h^r_{k-1}(u + \tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1}) - h^r_{k-1}(\tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1})\big] \end{cases}$$

Note that both terms of the first maximum are independent of Δ_k and both are decreasing in c_k; the second term is decreasing in c_k because of Lemma 3.8(1) (recall that $u \geq 0$). Now consider the second term in the second (subtracted) maximum. We can rewrite this as

$$\begin{aligned} &-\left(\frac{1-\delta^k}{1-\delta}(\Delta_k + u) - c_k\right) + \delta \mathbb{E}\big[h^r_{k-1}(u + \tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1}) \ - \ h^r_{k-1}(\tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1})\big] \\ &= -(1-\delta)\left(\frac{1}{1-\delta}(\Delta_k + u) - c_k\right) \\ &\quad - \ \delta \mathbb{E}\left[\frac{1-\delta^{k-1}}{1-\delta}(\Delta_k + u) - c_k + \ h^r_{k-1}(\tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1}) - h^r_{k-1}(u + \tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1})\right] \end{aligned}$$

Now note the reward here is CI-decreasing in (Δ_k, c_k) and, by the induction hypothesis, the continuation value is decreasing for changes in regions I or III. Thus the first maximum in (4) is increasing for changes in regions I or III and the second subtracted maximum in (4) is decreasing for such changes. Thus (4) and hence (3) is increasing for changes in regions I or III. □

Note that Lemma 3.8 can be stated in terms of h^r_k rather than v^r_k as follows. Let $\Delta^1_k \geq \Delta^2_k$. Then

(1) $h^r_k(\Delta^2_k, c_k) - h^r_k(\Delta^1_k, c_k)$ is increasing in c_k; and

(2) $h^r_k(\Delta^2_k, c_k) - h^r_k(\Delta^1_k, c_k) - c_k$ is decreasing in c_k.

h^r_k and v_k differ by a term that involves q_k but not c_k; see equation (5). Thus the two forms are equivalent. We will use the form involving h^r_k in the proof below.

Proof of Proposition 3.7(3). We prove this result by considering changes in regions I, II and III of Figure 3 separately.

First, consider changes in region I, with increases in Δ_k and decreases in c_k. In part (2) of Proposition 3.7, we showed that the policies are increasing in Δ_k for fixed c_k. We now show that with the additional assumption of non-decreasing quality, the policies are also decreasing in c_k for fixed p_k. Let $g_k(\Delta_k, c_k)$ be the difference between buying and waiting as defined in (1). We can rewrite this as

$$g_k(\Delta_k, c_k) \quad = \quad \frac{1-\delta^k}{1-\delta}\Delta_k - (1-\delta)c_k + \delta \mathbb{E}\big[h^r_{k-1}(\tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1}) - h^r_{k-1}(\Delta_k + \tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1}) - c_k\big] \, .$$

This is decreasing in c_k because, for each realization of $\tilde{u}^p_{k-1}$ and $\tilde{u}^c_{k-1}$, the expression inside the expectation is decreasing in c_k by Property (2) of Lemma 3.8. Thus $g_k(\Delta_k, c_k)$ is decreasing in c_k for fixed p_k as well as increasing p_k for fixed c_k. Thus if, it is optimal to adopt at (Δ^1_k, c^1_k) (i.e., $g_k(\Delta^1_k, c^1_k) \geq 0$), then it is also optimal to adopt at (Δ^2_k, c^2_k) (i.e., $g_k(\Delta^2_k, c^2_k) \geq 0$) when $(\Delta^2_k, c^2_k) \geq_{CI} (\Delta^1_k, c^1_k)$, $\Delta^2_k \geq \Delta^1_k$ and $c^2_k \leq c^1_k$, i.e., for changes in region I.

Now consider changes in region II, i.e., $(\Delta_k^2, c_k^2) \geq_{CI} (\Delta_k^1, c_k^1)$ and $c_k^2 \geq c_k^1$. Let $g_k(\Delta_k, c_k)$ be the difference between buying and waiting as defined in (1).

$$\begin{aligned}
g_k(\Delta_k^2, c_k^2) &= \frac{1-\delta^k}{1-\delta}\Delta_k^2 - c_k^2 + \delta\mathbb{E}\big[h_{k-1}^r(\tilde{u}_{k-1}^p, c_k^2 + \tilde{u}_{k-1}^c) - h_{k-1}^r(\Delta_k^2 + \tilde{u}_{k-1}^p, c_k^2 + \tilde{u}_{k-1}^c)\big] \\
&\geq \frac{1-\delta^k}{1-\delta}\Delta_k^2 - c_k^2 + \delta\mathbb{E}\big[h_{k-1}^r(\tilde{u}_{k-1}^p, c_k^1 + \tilde{u}_{k-1}^c) - h_{k-1}^r(\Delta_k^2 + \tilde{u}_{k-1}^p, c_k^1 + \tilde{u}_{k-1}^c)\big] \\
&= \Delta_k^2 - c_k^2 + \delta\mathbb{E}\left[h_{k-1}^r(\tilde{u}_{k-1}^p, c_k^1 + \tilde{u}_{k-1}^c) - \left(h_{k-1}^r(\Delta_k^2 + \tilde{u}_{k-1}^p, c_k^1 + \tilde{u}_{k-1}^c) + \frac{1-\delta^{k-1}}{1-\delta}\Delta_k^2\right)\right] \\
&\geq \Delta_k^1 - c_k^1 + \delta\mathbb{E}\left[h_{k-1}^r(\tilde{u}_{k-1}^p, c_k^1 + \tilde{u}_{k-1}^c) - \left(h_{k-1}^r(\Delta_k^2 + \tilde{u}_{k-1}^p, c_k^1 + \tilde{u}_{k-1}^c) + \frac{1-\delta^{k-1}}{1-\delta}\Delta_k^2\right)\right] \\
&\geq \Delta_k^1 - c_k^1 + \delta\mathbb{E}\left[h_{k-1}^r(\tilde{u}_{k-1}^p, c_k^1 + \tilde{u}_{k-1}^c) - \left(h_{k-1}^r(\Delta_k^1 + \tilde{u}_{k-1}^p, c_k^1 + \tilde{u}_{k-1}^c) + \frac{1-\delta^{k-1}}{1-\delta}\Delta_k^1\right)\right] \\
&= g_k(\Delta_k^1, c_k^1)
\end{aligned}$$

The first inequality follows from property (1) of Lemma 3.8, using the fact that $c_k^2 \geq c_k^1$ for changes in region II. The second inequality follows from the fact that $(\Delta_k^2, c_k^2) \geq_{CI} (\Delta_k^1, c_k^1)$, for changes in region II. The third inequality follows from Lemma A.2, using $\Delta_k^2 \geq \Delta_k^1$ for changes in region II. Thus if, it is optimal to adopt at (Δ_k^1, c_k^1) (i.e., $g_k(\Delta_k^1, c_k^1) \geq 0$), then it is also optimal to adopt at (Δ_k^2, c_k^2) (i.e., $g_k(\Delta_k^2, c_k^2) \geq 0$).

In region III, suppose $(\Delta_k^2, c_k^2) \geq_{CI} (\Delta_k^1, c_k^1)$ and $c_k^2 \leq c_k^1$.[1] We then have the following

$$\begin{aligned}
&g_k(\Delta_k^2, c_k^2) \\
&= \frac{1-\delta^k}{1-\delta}\Delta_k^2 - c_k^2 + \delta\mathbb{E}\big[h_{k-1}^r(\tilde{u}_{k-1}^p, c_k^2 + \tilde{u}_{k-1}^c) - h_{k-1}^r(\Delta_k^2 + \tilde{u}_{k-1}^p, c_k^2 + \tilde{u}_{k-1}^c)\big] \\
&= (1-\delta)(\frac{1}{1-\delta}\Delta_k^2 - c_k^2) + \delta\mathbb{E}\left[h_{k-1}^r(\tilde{u}_{k-1}^p, c_k^2 + \tilde{u}_{k-1}^c) - h_{k-1}^r(\Delta_k^2 + \tilde{u}_{k-1}^p, c_k^2 + \tilde{u}_{k-1}^c) - \frac{1-\delta^{k-1}}{1-\delta}\Delta_k^2 - c_k^2\right] \\
&\geq (1-\delta)(\frac{1}{1-\delta}\Delta_k^1 - c_k^1) + \delta\mathbb{E}\left[h_{k-1}^r(\tilde{u}_{k-1}^p, c_k^2 + \tilde{u}_{k-1}^c) - h_{k-1}^r(\Delta_k^2 + \tilde{u}_{k-1}^p, c_k^2 + \tilde{u}_{k-1}^c) - \frac{1-\delta^{k-1}}{1-\delta}\Delta_k^2 - c_k^2\right] \\
&\geq (1-\delta)(\frac{1}{1-\delta}\Delta_k^1 - c_k^1) + \delta\mathbb{E}\left[h_{k-1}^r(\tilde{u}_{k-1}^p, c_k^1 + \tilde{u}_{k-1}^c) - h_{k-1}^r(\Delta_k^1 + \tilde{u}_{k-1}^p, c_k^1 + \tilde{u}_{k-1}^c) - \frac{1-\delta^{k-1}}{1-\delta}\Delta_k^1 - c_k^1\right] \\
&= g_k(\Delta_k^1, c_k^1)
\end{aligned}$$

The first inequality follows from the assumption that $(\Delta_k^2, c_k^2) \geq_{CI} (\Delta_k^1, c_k^1)$ and the second inequality from Lemma A.3. Thus if, it is optimal to adopt at (Δ_k^1, c_k^1) (i.e., $g_k(\Delta_k^1, c_k^1) \geq 0$), then it is also optimal to adopt at (Δ_k^2, c_k^2) (i.e., $g_k(\Delta_k^2, c_k^2) \geq 0$). □

A.7. Proof of Proposition 3.8: Impact of Changing Costs

Proof of Lemma 3.8. We will work with v_k when establishing this result. We prove Lemma 3.8 using a joint induction argument on properties (1) and (2) of the lemma. Both properties hold trivially when $k = 0$. Assume that both properties (1) and (2) hold when there are $k-1$ periods to go. We first show that this implies property (1) holds for period k. By definition, we have

$$\begin{aligned}
v_k^r(p_k, c_k, q_k^2) - v_k^r(p_k, c_k, q_k^1) &= \max\begin{cases} p_k - c_k + \delta\mathbb{E}\big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, p_k)\big] \\ q_k^2 + \delta\mathbb{E}\big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, q_k^2)\big] \end{cases} \\
&\quad - \max\begin{cases} p_k - c_k + \delta\mathbb{E}\big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, q_k^1)\big] \\ q_k^1 + \delta\mathbb{E}\big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, q_k^1)\big] \end{cases}
\end{aligned}$$

[1] Note that this proof works for regions II and III, though the result for region II can be established much more easily.

If we subtract $q_k^1 + \delta\mathbb{E}\big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, q_k^1)\big]$ from each maximization statement the difference does not change and is equal to

$$= \max\left\{\begin{array}{l} \overbrace{p_k - c_k - q_k^1 + \delta\mathbb{E}\big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, p_k) - v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, q_k^1)\big]}^{x(c_k)} \\ \underbrace{q_k^2 - q_k^1 + \delta\mathbb{E}\big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, q_k^2) - v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, q_k^1)\big]}_{y(c_k)} \end{array}\right.$$

$$- \max\left\{\begin{array}{l} \overbrace{p_k - c_k - q_k^1 + \delta\mathbb{E}\big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, q_k^1) - v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, q_k^1)\big]}^{x(c_k)} \\ 0 \end{array}\right.$$

This difference is of the form $\max\{x(c_k), y(c_k)\} - \max\{x(c_k), 0\}$ where $x(c_k)$ and $y(c_k)$ have the following three conditions:

(i) $y(c_k) \geq 0$,

(ii) $x(c_k)$ is decreasing in c_k, and

(iii) $y(c_k)$ is increasing in c_k.

We show these conditions in a moment. Now depending on which terms take on the maxima,

$$\max\{x(c_k), y_2(c_k)\} - \max\{x(c_k), 0\} = \begin{cases} 0 & \text{if } x(c_k) \geq y(c_k) \text{ and } x(c_k) \geq 0, \\ y(c_k) - x(c_k) & \text{if } x(c_k) \leq y(c_k) \text{ and } x(c_k) \geq 0, \\ y(c_k) & \text{if } x(c_k) \leq y(c_k) \text{ and } x(c_k) \leq 0. \end{cases}$$

It is not possible to have $\max\{x(c_k), y_2(c_k)\} - \max\{x(c_k), 0\} = x(c_k)$ since $y(c_k) \geq 0$. In light of conditions (ii) and (iii), in each of the possible cases above, $\max\{x(c_k), y_2(c_k)\} - \max\{x(c_k), 0\}$ is increasing in c_k, which implies that Property (1) of Lemma 3.8 holds.

Now we show that conditions (i)-(iii) hold. Condition (i) above follows because $q_k^2 \geq q_k^1$ and $v_{k-1}^r(p_{k-1}, c_{k-1}, q_{k-1})$ is increasing in q_{k-1}. To establish condition (ii), we rewrite $x(c_k)$ as

$$\begin{aligned} x(c_k) = p_k - (1-\delta)c_k - q_k^1 + \delta\mathbb{E}\big[\tilde{u}_{k-1}^c\big] \\ + \delta\mathbb{E}\big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, p_k) - v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, q_k^1) - (c_k + \tilde{u}_{k-1}^c)\big] \end{aligned}$$

Note that the reward term here is decreasing in c_k. Also note that, for each $\tilde{u}_{k-1}^p$ and $\tilde{u}_{k-1}^c$, the term inside the expectation is decreasing in $c_k + \tilde{u}_{k-1}^c$ by the induction hypothesis for Property (2) of Lemma 3.8. Thus the expectations are decreasing in c_k. (To apply the induction hypothesis here we use the fact that the technology is almost certainly improving over time to ensure that $p_k \geq q_k^1$.) Thus $x(c_k)$ is decreasing in c_k.

To establish condition (iii), recall

$$y(c_k) = q_k^2 - q_k^1 + \delta\mathbb{E}\big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, q_k^2) - v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, q_k^1)\big]\,.$$

Note that, for each $\tilde{u}_{k-1}^p$ and $\tilde{u}_{k-1}^c$, the term inside the expectation is increasing in $c_k + \tilde{u}_{k-1}^c$ by the induction hypothesis for Property (1) of Lemma 3.8. (Here we use the assumption that $q_k^2 \geq q_k^1$ to apply the induction hypothesis.) Thus the expectation and hence $y(c_k)$ is increasing in c_k. This completes the proof of Property (1) of Lemma 3.8.

We next prove Property (2) of Lemma 3.8. Consider

$$\begin{aligned} & v_k^r(p_k, c_k, q_k^2) - v_k^r(p_k, c_k, q_k^1) - c_k \\ & \quad = \max\left\{\begin{array}{l} p_k - 2c_k + \delta\mathbb{E}\big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, p_k)\big] \\ q_k^2 - c_k + \delta\mathbb{E}\big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, q_k^2)\big] \end{array}\right. \\ & \quad\quad - \max\left\{\begin{array}{l} p_k - c_k + \delta\mathbb{E}\big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, p_k)\big] \\ q_k^1 + \delta\mathbb{E}\big[v_{k-1}^r(p_k + \tilde{u}_{k-1}^p, c_k + \tilde{u}_{k-1}^c, q_k^1)\big] \end{array}\right. \end{aligned}$$

If we subtract $p_k - c_k + \delta\mathbb{E}\big[v^r_{k-1}(p_k + \tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1}, p_k)\big]$ from each maximization statement the difference does not change and is equal to

$$
\begin{aligned}
& v^r_k(p_k, c_k, q^2_k) - v^r_k(p_k, c_k, q^1_k) - c_k \\
&\quad = \max\left\{ \begin{array}{l} -c_k \\ q^2_k - p_k + \delta\mathbb{E}\big[v^r_{k-1}(p_k + \tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1}, q^2_k) - v^r_{k-1}(p_k + \tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1}, p_k)\big] \end{array}\right. \\
&\qquad - \max\left\{ \begin{array}{l} 0 \\ q^1_k - p_k + c_k + \delta\mathbb{E}\big[v^r_{k-1}(p_k + \tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1}, q^1_k) - v^r_{k-1}(p_k + \tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1}, p_k)\big] \end{array}\right. \\
&\quad = \max\left\{ \begin{array}{l} -c_k \\ q^2_k - p_k + \delta\mathbb{E}\big[v^r_{k-1}(p_k + \tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1}, q^2_k) - v^r_{k-1}(p_k + \tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1}, p_k)\big] \end{array}\right. \\
&\qquad - \max\left\{ \begin{array}{l} 0 \\ q^1_k - p_k + (1-\delta)c_k - \delta\mathbb{E}\big[\tilde{u}^c_{k-1}\big] \\ \qquad +\delta\mathbb{E}\big[v^r_{k-1}(p_k + \tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1}, q^1_k) - v^r_{k-1}(p_k + \tilde{u}^p_{k-1}, c_k + \tilde{u}^c_{k-1}, p_k) + c_k + \tilde{u}^c_{k-1}\big] \end{array}\right.
\end{aligned}
$$

The first maximization statement is decreasing in c_k using the induction hypothesis on Property (1); note that we use $p_k \geq q^2_k$ because the technology is almost certainly improving. The second maximization statement is increasing in c_k using the induction hypothesis on Property (2); here we again use the assumption that the technology is almost certainly improving to conclude that $p_k \geq q^1_k$. Then, the difference above is decreasing in c_k. □

A.8. Alternative Partial Orders

Though we have focused on improvements defined in terms of the CI-order, we can establish similar monotonicity results using other partial orders on the technologies instead. These alternative orders may allow us to use essentially the same arguments as above, but with transitions that are not CI-increasing or do not exhibit CI-diminishing returns. To use these arguments, the alternative partial order $\succcurlyeq$ must be "stronger" than the CI-order in that $(p^2_k, c^2_k) \succcurlyeq (p^1_k, c^1_k)$ implies $(p^2_k, c^2_k) \geq_{CI} (p^1_k, c^1_k)$: this ensures that the rewards associated with adoption will be $\succcurlyeq$-increasing in all three models. We can then define increasing functions, increasing transitions, and diminishing improvements as in Definitions 3.2 and 3.5 using the $\succcurlyeq$-order in place of the CI-order. Then using the same proofs that we use with the CI-order in Proposition 3.4, we can then show that $\succcurlyeq$-increasing transitions lead to $\succcurlyeq$-increasing value functions in all three models. Similarly, if we have $\succcurlyeq$-increasing transitions that exhibit $\succcurlyeq$-diminishing improvements, we can use the proof of Proposition 3.6 to show that the optimal policies in the single-purchase model are $\succcurlyeq$-increasing.

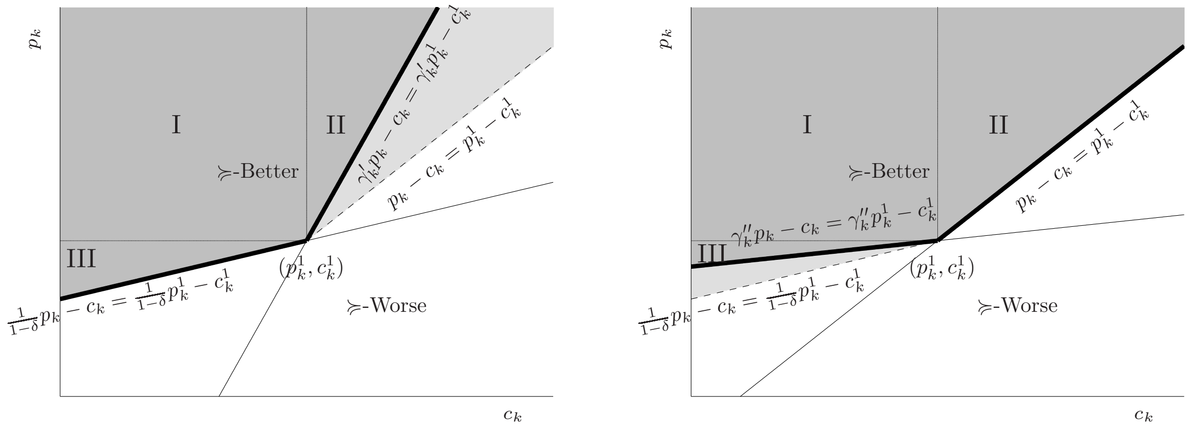
For example, consider a generalized additive model of transitions of the form

$$
\begin{aligned}
\tilde{p}_{k-1} &= \beta_p p_k + \tilde{u}^p_{k-1} \\
\tilde{c}_{k-1} &= \beta_c c_k + \tilde{u}^c_{k-1}
\end{aligned} \tag{5}
$$

where $\tilde{u}^p_{k-1}$ and $\tilde{u}^c_{k-1}$ may be correlated but are independent of p_k and c_k; we will assume that $\beta_p, \beta_c \geq 0$. If $\beta_p = \beta_c = 1$, the technology transitions are of the additive form of equation (1). In a model of the form of (5), the expected increment in quality is $\mathbb{E}[\tilde{p}_{k-1}] - p_k = (\beta_p - 1)p_k + \mathbb{E}\big[\tilde{u}^p_{k-1}\big]$, so a coefficient $\beta_p < 1$ corresponds to a case where the expected improvements in quality are linearly decreasing in the quality level p_k. This might be a reasonable model if improvements in quality make it harder to find further improvements. Also note that if $\beta_p < 1$, the quality levels will converge over time and the initial quality level p_k has little influence on the quality level in the distant future. The cost equation has a similar interpretation.

The transitions of the generalized additive form of (5) will be CI-increasing if $\beta_p = \beta_c$ and will exhibit CI-diminishing improvements if $\beta_p = \beta_c$ and $\beta_p, \beta_c \leq 1$. However the generalized additive transitions will not be CI-increasing or exhibit CI-diminishing improvements if $\beta_p \neq \beta_c$. We can however establish monotonicity properties for other coefficients in equation (5) and other forms of transitions by using the same arguments as before, but with a different partial order on technologies. Specifically, given a generalized additive model of the form of equation (5) and its coefficients β_p and β_c, we can define a partial order $\succcurlyeq_{(\beta_p,\beta_c)}$ such that $(p^2_k, c^2_k) \succcurlyeq_{(\beta_p,\beta_c)} (p^1_k, c^1_k)$ if

$$
\gamma_k p^1_k - c^1_k \leq \gamma_k p^2_k - c^2_k, \tag{6}
$$



(a) $\beta_c > \beta_p, \gamma'_k = \left(\frac{\beta_p}{\beta_c}\right)^{k-1}$

(b) $\beta_c < \beta_p, \gamma''_k = \left(\frac{\beta_p}{\beta_c}\right)^{k-1} \frac{1}{1-\delta}$

Figure 1: (p_k, c_k) that are better or worse than (p^1_k, c^1_k) under the $\succcurlyeq_{(\beta_p,\beta_c)}$-order.

for all γ_k such that

$$\min\left\{1, \left(\frac{\beta_p}{\beta_c}\right)^{k-1}\right\} \leq \gamma_k \leq \max\left\{1, \left(\frac{\beta_p}{\beta_c}\right)^{k-1}\right\} \frac{1}{1-\delta}. \tag{7}$$

If $\beta_c = \beta_p$, this order is equivalent to the CI-order and in other cases it is stronger than the CI-order. Note that the $\succcurlyeq_{(\beta_p,\beta_c)}$-order implicitly depends on k; the appropriate k will be clear in the context where it is used, e.g., when comparing (p^2_k, c^2_k) and (p^1_k, c^1_k).

The sets of $\succcurlyeq_{(\beta_p,\beta_c)}$-improvements are shown in Figure 1, with the case where $\beta_c > \beta_p$ shown in Figure 1(a) and the case $\beta_c < \beta_p$ shown in Figure 1(b). If $\beta_c > \beta_p$, then the minimum on the left of (7) is achieved by $(\beta_p/\beta_c)^{k-1}$ and the maximum on the right is achieved by 1. Graphically in terms of the regions of Figure 3, in this case the changes in regions I and III are $\succcurlyeq_{(\beta_p,\beta_c)}$-improvements, though some of the changes in region II are not: there is a wedge of cases in the upper right quadrant in Figure 1(a) that are CI-improvements but are not $\succcurlyeq_{(\beta_p,\beta_c)}$-improvements. For the changes in this wedge, quality and cost both increase with the quality increase being sufficient to ensure that the new technology is preferred over the old even if the technology were held only for a single period. Though such changes would benefit the consumer who is adopting, because the impact of changes in costs diminsh more slowly than changes in quality (because $\beta_c > \beta_p$), we cannot be sure that these changes will benefit a consumer who is waiting. Note that as k grows large, $(\beta_p/\beta_c)^{k-1}$ approaches zero and, asymptotically, no changes in region II are $\succcurlyeq_{(\beta_p,\beta_c)}$-improvements.

On the other hand, if $\beta_c < \beta_p$, then the minimum on the left is achieved by 1 and the maximum on the right is achieved by $(\beta_p/\beta_c)^{k-1}$. In this case, changes in regions I and II are $\succcurlyeq_{(\beta_p,\beta_c)}$-improvements, but some of the changes in region III are not. There is a wedge of these in the lower left quadrant in Figure 1(b) that are CI-improvements but are not $\succcurlyeq_{(\beta_p,\beta_c)}$-improvements. As in the previous case, changes in this wedge would be preferred if the technology were adopted but we cannot be sure that such changes will benefit a consumer who is waiting. Also, as k grows large, $(\beta_p/\beta_c)^{k-1}$ approaches infinity and, asymptotically, no changes in region III are $\succcurlyeq_{(\beta_p,\beta_c)}$-improvements.

We can show that the transitions of the generalized additive model of equation (5) are $\succcurlyeq_{(\beta_p,\beta_c)}$-increasing (see Appendix A.9 for a proof). This imples that with transitions of this form, the single-purchase and repeat-purchase value functions will be $\succcurlyeq_{(\beta_p,\beta_c)}$-increasing; i.e., $v^s_k(p_k, c_k, q_k)$ and $v^r_k(p_k, c_k, q_k)$ will be increasing in directions of $\succcurlyeq_{(\beta_p,\beta_c)}$-improvements shown in Figure 1.

We can similarly establish monotonicity of the policies for the single-purchase model. The transitions of the generalized additive model of equation (5) exhibit $\succcurlyeq_{(\beta_p,\beta_c)}$-diminishing improvements if $\beta_c \leq \beta_p \leq 1$ (see Appendix A.9 for a proof). Thus, in these cases, the policies for the single-purchase model will be

$\succcurlyeq_{(\beta_p,\beta_c)}$-increasing. As discussed following equation (5), coefficients β_p and β_c less than one imply that the expected improvements in quality and costs are linearly decreasing in the quality level p_k and cost level c_k. The condition $\beta_c \leq \beta_p$ required for $\succcurlyeq_{(\beta_p,\beta_c)}$-diminishing improvements means the expected improvements in costs are decreasing at a faster rate than the expected improvements in quality. If $\beta_c > \beta_p$, the compounding in the generalized additive model may cause the value of waiting to increase more than value of adopting and destroy the monotonicity of the policies, even in the single-purchase model.

We do not hold much hope for establishing monotonic policies in the repeat-purchase model with generalized additive transitions. For instance in the first example of §3.3, we have a generalized additive model for transitions with $\beta_p = \beta_c = 0.7$ and we have non-monotonic policies even as we change price alone. The additive structure of the transitions (i.e., the assumption that $\beta_p = \beta_c = 1.0$) underlies the simplification of the value function in equations (5) and (6) and is required in the proof of Proposition 3.7.

A.9. Proofs for Alternative Orders

Proof that generalized additive transitions are $\succcurlyeq_{(\beta_p,\beta_c)}$-increasing. We will show that for any $k \geq 1$ the generalized additive transitions of equations (5) are $\succcurlyeq_{(\beta_p,\beta_c)}$-increasing, that is $(p_k^2, c_k^2) \succcurlyeq_{(\beta_p,\beta_c)} (p_k^1, c_k^1)$ implies that $(\tilde{p}_{k-1}\tilde{c}_{k-1})|(p_k^2, c_k^2) \tilde{\succcurlyeq}_{(\beta_p,\beta_c)} (\tilde{p}_{k-1}\tilde{c}_{k-1})|(p_k^1, c_k^1)$ where $\tilde{\succcurlyeq}_{(\beta_p,\beta_c)}$ denotes the (first-order) stochastic dominance relation defined by the $\succcurlyeq_{(\beta_p,\beta_c)}$-partial order; see Definition 3.2.

Using Proposition 3.3(4), we can show that $(\tilde{p}_{k-1}\tilde{c}_{k-1})|(p_k^2, c_k^2) \tilde{\succcurlyeq}_{(\beta_p,\beta_c)} (\tilde{p}_{k-1}\tilde{c}_{k-1})|(p_k^1, c_k^1)$ by showing that

$$(\tilde{p}_{k-1}\tilde{c}_{k-1})|(p_k^1, c_k^1) \succcurlyeq_{(\beta_p,\beta_c)} (\tilde{p}_{k-1}\tilde{c}_{k-1})|(p_k^2, c_k^2) \tag{8}$$

holds almost certainly. Using the definition of the $\succcurlyeq_{(\beta_p,\beta_c)}$-order and the definition of the additive transitions (5), this is equivalent to showing that

$$\gamma_{k-1}(\beta_p p_k^1 + \tilde{u}_{k-1}^p) - (\beta_c c_k^1 + \tilde{u}_{k-1}^c) \quad \leq \quad \gamma_{k-1}(\beta_p p_k^2 + \tilde{u}_{k-1}^p) - (\beta_c c_k^2 + \tilde{u}_{k-1}^c). \tag{9}$$

holds for all γ_{k-1} such that

$$\min\left\{1, \left(\frac{\beta_p}{\beta_c}\right)^{k-2}\right\} \leq \gamma_{k-1} \leq \max\left\{\frac{1}{1-\delta}, \left(\frac{\beta_p}{\beta_c}\right)^{k-2} \frac{1}{1-\delta}\right\} . \tag{10}$$

Canceling common terms and dividing through by $\beta_c > 0$, (9) is equivalent to

$$\gamma_{k-1}\frac{\beta_p}{\beta_c}p_k^1 - c_k^1 \quad \leq \quad \gamma_{k-1}\frac{\beta_p}{\beta_c}p_k^2 - c_k^2. \tag{11}$$

Taking $\gamma_k = \gamma_{k-1}\frac{\beta_p}{\beta_c}$, showing (9) and (10) becomes equivalent to requiring

$$\gamma_k p_k^1 - c_k^1 \leq \gamma_k p_k^2 - c_k^2, \tag{12}$$

for all γ_k such that

$$\min\left\{\frac{\beta_p}{\beta_c}, \left(\frac{\beta_p}{\beta_c}\right)^{k-1}\right\} \leq \gamma_k \leq \max\left\{\frac{\beta_p}{\beta_c}\frac{1}{1-\delta}, \left(\frac{\beta_p}{\beta_c}\right)^{k-1} \frac{1}{1-\delta}\right\}. \tag{13}$$

We now show that $(p_k^2, c_k^2) \succcurlyeq_{(\beta_p,\beta_c)} (p_k^1, c_k^1)$ implies that (9) holds for all γ_{k-1} satisfying (10) or, equivalently (12) holds for all γ_k satisfying (13). $(p_k^2, c_k^2) \succcurlyeq_{(\beta_p,\beta_c)} (p_k^1, c_k^1)$ means (6) holds for all γ_k satisfying (7). Note (12) and (6) are identical. So we must show that if (9)=(6) holds for all γ_k satisfying (7), (12)=(6) holds for all γ_k satisfying (13). To see this note that for any $\beta_p, \beta_c > 0$,

$$\text{LHS of (7)} = \min\left\{1, \left(\frac{\beta_p}{\beta_c}\right)^{k-1}\right\} \leq \min\left\{\frac{\beta_p}{\beta_c}, \left(\frac{\beta_p}{\beta_c}\right)^{k-1}\right\} = \text{LHS of (13)}$$

and

$$\text{RHS of (13)} = \max\left\{\frac{\beta_p}{\beta_c}\frac{1}{1-\delta}, \left(\frac{\beta_p}{\beta_c}\right)^{k-1}\frac{1}{1-\delta}\right\} \le \max\left\{\frac{1}{1-\delta}, \left(\frac{\beta_p}{\beta_c}\right)^{k-1}\frac{1}{1-\delta}\right\} = \text{RHS of (7)} .$$

Thus, the range of values of γ_k considered in (13) is narrower than that considered in (7).

Thus if $(p_k^2, c_k^2) \succcurlyeq_{(\beta_p,\beta_c)} (p_k^1, c_k^1)$ then (9) holds for all γ_{k-1} satisfying (10); that is, the generalized additive transitions will be $\succcurlyeq_{(\beta_p,\beta_c)}$-increasing. □

Proof that generalized additive transitions exhibit $\succcurlyeq_{(\beta_p,\beta_c)}$-diminishing improvements. We now show that the generalized additive transitions exhibit $\succcurlyeq_{(\beta_p,\beta_c)}$-diminishing improvements for $0 < \beta_c \le \beta_p \le 1$. With the additive transitions, the condition for diminishing improvements can be written as:

$$\begin{aligned}
&\delta\mathbb{E}\Big[\frac{1-\delta^{k-1}}{1-\delta}\tilde{p}_{k-1} - \tilde{c}_{k-1}|p_k, c_k\Big] - \left(\frac{1-\delta^k}{1-\delta}p_k - c_k\right) \\
&= \left(\delta\frac{1-\delta^{k-1}}{1-\delta}\beta_p - \frac{1-\delta^k}{1-\delta}\right)p_k - (\delta\beta_c - 1)c_k + \delta\mathbb{E}\Big[\frac{1-\delta^{k-1}}{1-\delta}\tilde{u^p}_{k-1} - \tilde{u^c}_{k-1}\Big] \\
&= -\left[\left(\beta_p + \frac{1-\delta^k}{1-\delta}(1-\beta_p)\right)p_k - (1-\beta_c + (1-\delta)\beta_c)\,c_k\right] + \delta\mathbb{E}\Big[\frac{1-\delta^{k-1}}{1-\delta}\tilde{u^p}_{k-1} - \tilde{u^c}_{k-1}\Big] \\
&= -\,(1-\beta_c + (1-\delta)\beta_c)\left[\frac{\beta_p + \frac{1-\delta^k}{1-\delta}(1-\beta_p)}{1-\beta_c+(1-\delta)\beta_c}p_k - c_k\right] + \delta\mathbb{E}\Big[\frac{1-\delta^{k-1}}{1-\delta}\tilde{u^p}_{k-1} - \tilde{u^c}_{k-1}\Big]
\end{aligned}$$

To show that the generalized additive transitions have $\succcurlyeq_{(\beta_p,\beta_c)}$-diminishing improvements, we must show that this expression is $\succcurlyeq_{(\beta_p,\beta_c)}$-decreasing, or equivalently that the expression in the brackets above,

$$\frac{\beta_p + \frac{1-\delta^k}{1-\delta}(1-\beta_p)}{1-\beta_c+(1-\delta)\beta_c}p_k - c_k, \tag{14}$$

is a $\succcurlyeq_{(\beta_p,\beta_c)}$-increasing function. Notice that this function is of the form $\gamma_k' p_k - c_k$ where

$$\gamma_k' = \frac{\beta_p + \frac{1-\delta^k}{1-\delta}(1-\beta_p)}{1-\beta_c+(1-\delta)\beta_c}.$$

Thus, using the definition of the $\succcurlyeq_{(\beta_p,\beta_c)}$-order, to show that (14) is a $\succcurlyeq_{(\beta_p,\beta_c)}$-increasing function, it is enough to show that we have

$$1 \le \gamma_k' \le \left(\frac{\beta_p}{\beta_c}\right)^{k-1}\frac{1}{1-\delta}. \tag{15}$$

Because, if (15) holds, than by the definition of $\succcurlyeq_{(\beta_p,\beta_c)}$-order, (14) is a $\succcurlyeq_{(\beta_p,\beta_c)}$-increasing function.

Showing $1 \le \gamma_k'$ is equivalent to proving

$$1-\beta_c+(1-\delta)\beta_c \le \beta_p + \frac{1-\delta^k}{1-\delta}(1-\beta_p)$$

Because we have $0 < \beta_c \le \beta_p \le 1$, the LHS is a convex combination of $(1-\delta)$ (with weight β_c) and 1 (with weight $(1-\beta_c)$) and hence lies between $1-\delta$ and 1. The RHS is a convex combination of 1 (with weight β_p) and $\frac{1-\delta^k}{1-\delta}$ (with weight $(1-\beta_p)$). Since $\delta \le 1 \le \frac{1-\delta^k}{1-\delta}$, we have LHS $\le$ RHS.

We next show that $\gamma_k' \le \frac{1}{1-\delta} \le \left(\frac{\beta_p}{\beta_c}\right)^{k-1}\frac{1}{1-\delta}$ which implies the second inequality because $0 < \beta_c \le \beta_p$. Proving $\gamma_k' \le \frac{1}{1-\delta}$ is equivalent to proving

$$\beta_p + \frac{1-\delta^k}{1-\delta}(1-\beta_p) \le (1-\beta_c)\frac{1}{1-\delta} + \beta_c, \tag{16}$$

which follows if we have

$$\beta_p + \frac{1-\delta^k}{1-\delta}(1-\beta_p) \le (1-\beta_c)\frac{1-\delta^k}{1-\delta} + \beta_c, \tag{17}$$

because RHS of (17) is less than or equal to the RHS of (16) for $\beta_c \le 1$. Notice that both the LHS and the RHS are convex combinations of 1 and $\frac{1-\delta^k}{1-\delta}$. Because $\beta_c \le \beta_p$, the RHS of (17) puts more weight on the larger value ($\frac{1-\delta^k}{1-\delta}$) than the LHS and hence the inequality holds. □

A.10. Proof of Proposition 4.2

Proof.

1. The single-purchase value function for $k=0$ is trivially convex. Assume that $v^s_{k-1}(p_{k-1}, c_{k-1}, q_{k-1})$ is convex in (p_{k-1}, c_{k-1}). The value from adopting when there are k periods to go, $\frac{1-\delta^k}{1-\delta}p_k - c_k$, is convex in (p_k, c_k). The value from waiting is also convex because v^s_{k-1} is convex (by the induction hypothesis) and transitions are convex. Because the maximum of two convex functions is convex, the single-purchase value function when there are k periods to go must be convex.

2. The value from adopting technology (p^α_k, c^α_k) is $\frac{1-\delta^k}{1-\delta}p^\alpha_k - c^\alpha_k$.

$$\begin{aligned}\frac{1-\delta^k}{1-\delta}p^\alpha_k - c^\alpha_k &= \alpha\left(\frac{1-\delta^k}{1-\delta}p^1_k - c^1_k\right) + (1-\alpha)\left(\frac{1-\delta^k}{1-\delta}p^2_k - c^2_k\right)\\ &\ge \alpha\left(q_k + \delta\mathbb{E}\big[v^s_{k-1}(\tilde{p}_{k-1}, \tilde{c}_{k-1}, q_k)|p^1_k, c^1_k\big]\right) + (1-\alpha)\left(q_k + \delta\mathbb{E}\big[v^s_{k-1}(\tilde{p}_{k-1}, \tilde{c}_{k-1}, q_k)|p^2_k, c^2_k\big]\right)\\ &\ge q_k + \delta\mathbb{E}\big[v^s_{k-1}(\tilde{p}_{k-1}, \tilde{c}_{k-1}, q_k)|p^\alpha_k, c^\alpha_k\big]\end{aligned}$$

The first inequality follows because it is optimal to adopt with both (p^1_k, c^1_k) and (p^2_k, c^2_k); the second inequality follows because single-purchase value function is convex and technology transitions are convex.

□

A.11. Proof of Proposition 4.3

We need the following lemma to prove that the repeat-purchase value function is convex in (p_k, c_k, q_k).

Lemma A.4. *If the transitions are as in* (5)*, then* $\mathbb{E}[u(\tilde{p}_{k-1}, \tilde{c}_{k-1}, p_k)|p_k, c_k]$ *and* $\mathbb{E}[u(\tilde{p}_{k-1}, \tilde{c}_{k-1}, q_k)|p_k, c_k]$ *are convex in* p_k, c_k *and* q_k *for convex* u.

Proof of Lemma A.4. We will prove that $\mathbb{E}[u(\tilde{p}_{k-1}, \tilde{c}_{k-1}, p_k)|p_k, c_k]$ is convex in p_k and c_k. Proving that $\mathbb{E}[u(\tilde{p}_{k-1}, \tilde{c}_{k-1}, q_k)|p_k, c_k]$ is convex in p_k, c_k and q_k is similar. Because transitions satisfy (5), we have

$$\mathbb{E}[u(\tilde{p}_{k-1}, \tilde{c}_{k-1}, p^\alpha_k)|p^\alpha_k, c^\alpha_k] = \mathbb{E}\big[u(\beta_p p^\alpha_k + \tilde{u}^p_{k-1}, \beta_c c^\alpha_k + \tilde{u}^c_{k-1}, p^\alpha_k)\big]$$

Because u is convex, for each realization of $\tilde{u}^p_{k-1}$ and $\tilde{u}^c_{k-1}$, we have

$$u(\beta_p p^\alpha_k + u^p_{k-1}, \beta_c c^\alpha_k + u^c_{k-1}, p^\alpha_k) \le \alpha u(\beta_p p^1_k + u^p_{k-1}, \beta_c c^1_k + u^c_{k-1}, p^1_k) + (1-\alpha)u(\beta_p p^2_k + u^p_{k-1}, \beta_c c^2_k + u^c_{k-1}, p^2_k)$$

Then, it must also hold in expectation and we have

$$\begin{aligned}\mathbb{E}[u(\tilde{p}_{k-1}, \tilde{c}_{k-1}, p^\alpha_k)|p^\alpha_k, c^\alpha_k] &\le \alpha\mathbb{E}\big[u(\beta_p p^1_k + \tilde{u}^p_{k-1}, \beta_c c^1_k + \tilde{u}^c_{k-1}, p^1_k)\big] + (1-\alpha)\mathbb{E}\big[u(\beta_p p^2_k + \tilde{u}^p_{k-1}, \beta_c c^2_k + \tilde{u}^c_{k-1}, p^2_k)\big]\\ &= \mathbb{E}\big[u(\tilde{p}_{k-1}, \tilde{c}_{k-1}, p^1_k)|p^1_k, c^1_k\big] + (1-\alpha)\mathbb{E}\big[u(\tilde{p}_{k-1}, \tilde{c}_{k-1}, p^2_k)|p^2_k, c^2_k\big]\end{aligned}$$

□

Proof of Proposition 4.3. To show that the repeat-purchase value function is convex, we will proceed by induction. The property holds trivially for $k=0$. Assume $v^r_{k-1}(p_{k-1}, c_{k-1}, q_{k-1})$ is convex in p_{k-1}, c_{k-1} and q_{k-1}. Then,

$$v^r_{k-1}(p_k, c_k, q_k) = \max\begin{cases} p_k - c_k + \delta\mathbb{E}\big[v^r_{k-1}(\tilde{p}_{k-1}, \tilde{c}_{k-1}, p_k)|p_k, c_k\big]\\ q_k + \delta\mathbb{E}\big[v^r_{k-1}(\tilde{p}_{k-1}, \tilde{c}_{k-1}, q_k)|p_k, c_k\big]\end{cases}$$

The rewards are convex in p_k, c_k and q_k and the continuation values are convex following Lemma A.4, then, the value function must be convex because the maximum of two convex functions is convex. $\square$