B. Online Appendix

B.1. Constructing examples with nonmonotonic adoption policies

Assume $c > 0$ and the utility function $u(w)$ is increasing and approaches $-\infty$ as w approaches 0. Suppose we have a prior distribution π_2, initial wealth w_0, and a likelihood function $L(x|\theta)$ (with $L(x|\theta) > 0$ for all x and θ) satisfying the MLR property such that waiting is strictly preferred to adopting and adopting is strictly preferred to quitting. We will now construct a π_1 and likelihood function $L^*(x|\theta)$ (satisfying the MLR property) such that $\pi_2 \succeq_{LR} \pi_1$ and it is optimal to adopt with π_1 but not with π_2.

Adopting being strictly preferred to quitting with π_2 implies

$$s_1 \equiv \mathbb{E}[\, u(w_0 + \tilde{\theta}) \,|\, \pi_2 \,] - u(w_0) > 0 \ . \tag{B1}$$

Let θ_m denote the minimum point of support for π_2. Note $\theta_m < 0$, otherwise it would be optimal to adopt immediately.

We now construct a new prior π_1 such that $\pi_2 \succeq_{LR} \pi_1$ and adopting is optimal with π_1. Let $\theta_\ell = -w + c$ and π_1 be a new prior with mass p_ℓ $(0 < p_\ell < 1)$ at θ_ℓ is mixed with $(1 - p_\ell)$ times π_2; p_ℓ will be specified shortly. With this construction, $\pi_2 \succeq_{LR} \pi_1$. Let L^* be an augmented likelihood function where we take $L^*(x|\theta_\ell) = L(x|\theta_m)$ and $L^*(x|\theta) = L(x|\theta)$ for all other θ and x. Since $L(x|\theta_m)$ is assumed to satisfy the MLR property, $L^*(x|\theta_m)$ does as well.

We want to choose p_ℓ such that adopting is preferred to quitting, i.e., such that

$$\mathbb{E}[\, u(w_0 + \tilde{\theta}) \,|\, \pi_1 \,] - u(w_0) = \epsilon$$

where ϵ may be arbitrarily chosen to satisfy $0 < \epsilon < s_1$ and s_1 is defined in (B1). This can be rewritten as

$$\begin{aligned} & p_\ell \left(u(w_0 + \theta_\ell) - u(w_0) \right) + (1 - p_\ell) \left(\mathbb{E}[\, u(w_0 + \tilde{\theta}) \,|\, \pi_2 \,] - u(w_0) \right) \\ = \ & p_\ell \left(u(w_0 + \theta_\ell) - u(w_0) \right) + (1 - p_\ell) s_1 \ = \ \epsilon. \end{aligned}$$

Solving for p_ℓ, we have

$$p_\ell = \frac{s_1 - \epsilon}{s_1 - (u(w_0 + \theta_\ell) - u(w_0))} \ .$$

The resulting p_ℓ will satisfy $0 < p_\ell < 1$ (because $s_1 > 0$ and $\theta_\ell < 0$).

Because $p_\ell > 0$ and $L^*(x|\theta) > 0$ for all x and θ, the posterior probability $\Pi(\theta_\ell; x, \pi_1)$ must be positive for all signals x. Since $u(w + \theta_\ell - c) = -\infty$, the expected utility associated with waiting and then adopting must also be $-\infty$, for each signal. Thus it cannot be optimal to wait with π_1.

Therefore, we have constructed an example of a nonmonotonic policy: with this π_1 and π_2 and likelihood function $L^*(x|\theta_m)$, we have $\pi_2 \succeq_{LR} \pi_1$ and it is optimal to adopt with π_1 but not with π_2.

B.2. Comment on Proposition 6.2(ii)

The result of Proposition 6.2(ii) generalizes to the setting where the underlying state θ (here the benefit of technology) is changing over time (as in a partially observed Markov decision process), provided the state transitions satisfy the MLR property. Let θ_k denote the state variable in period k and let $\nu(\theta_{k-1}|\theta_k)$ denote the transition probabilities for θ_{k-1} conditional on the current period state θ_k. The prior on next period's state is then:

$$\eta(\theta_{k-1}; \pi) = \int_{\theta_k} \nu(\theta_{k-1}|\theta_k) \pi(\theta_k) \, d\theta_k$$

and the signal and posterior distributions are then

$$f(x; \pi) = \int_{\theta_{k-1}} L(x|\theta_{k-1}) \eta(\theta_{k-1}; \pi) \, d\theta_{k-1} \text{ and}$$

$$\Pi(\theta_{k-1};\pi,x) = \frac{L(x|\theta_{k-1})\eta(\theta_{k-1};\pi)}{f(x;\pi)} \ .$$

If we assume that the technology transitions, as well as the signal process, satisfy the MLR property: $\nu(\theta_{k-1}|\theta_k^2) \succeq_{LR} \nu(\theta_{k-1}|\theta_k^1)$ for all $\theta_k^2 \geq \theta_k^1$ and k, then the next-period prior $\eta(\pi)$, predictive distribution for signals $f(\pi)$, and posteriors $\Pi(\pi,x)$ are all LR improving with LR improvements in the prior π for the current period. These results follow from Proposition 3.1. The proof of Proposition 6.2(ii) proceeds as before, except in (A2) (and the preceding inequality) we have the prior on the next period state $\eta(\theta;\pi_i)$ in place of $\pi_i(\theta)$; the same argument then applies.

B.3. Proof of Proposition 8.2

In this section, we focus on the case where the utility function is an exponential $u(w) = -\exp(-w/R)$ and define the certainty equivalent as $\mathrm{CE}(u) = -R\ln(-u)$. Also recall the "delta property" for exponential utilities: $\mathrm{CE}(\,\mathbb{E}[\,u(\tilde\theta+\Delta)\,]) = \mathrm{CE}(\,\mathbb{E}[\,u(\tilde\theta)\,]) + \Delta$.

We prove Proposition 8.2 with the aid of the following lemma.

Lemma B.1. *With an exponential utility function and a signal process that satisfies the MLR property,*

$$\mathrm{CE}(\,\mathbb{E}[\,u(w+\tilde\theta)\,|\,\pi\,]) - \mathrm{CE}(U_k(w,\pi))$$

is LR-increasing.

Proof. Note that

$$\begin{aligned}
&\mathrm{CE}(\,\mathbb{E}[\,u(w+\tilde\theta)\,|\,\pi\,]) - \mathrm{CE}(U_k(w,\pi))\\
&= \min\begin{cases} 0 & \text{(adopt)}\\ \mathrm{CE}(\,\mathbb{E}[\,u(w+\tilde\theta)\,|\,\pi\,]) - \mathrm{CE}(\,\mathbb{E}[\,U_{k-1}(w-c,\Pi(\pi,\tilde x))\,|\,f(\pi)\,]) & \text{(wait)}\\ \mathrm{CE}(\,\mathbb{E}[\,u(w+\tilde\theta)\,|\,\pi\,]) - w & \text{(reject)}\end{cases} \ .
\end{aligned}$$

The reject term here, $\mathrm{CE}(\,\mathbb{E}[\,u(w+\tilde\theta)\,|\,\pi\,]) - w$, is LR-increasing because $\mathbb{E}[\,u(w+\tilde\theta)\,|\,\pi\,]$ is LR-increasing and $\mathrm{CE}(u)$ is an increasing function. The adoption term (0) is trivially LR-increasing. Because minimum of three LR-increasing functions is LR-increasing, we can complete the proof by showing the wait term above is also LR-increasing.

From Proposition 7.2, we know that $\mathbb{E}[\,u(w+\tilde\theta-c)\,|\,\pi\,] - \mathbb{E}[\,U_{k-1}(w-c,\Pi(\pi,\tilde x))\,|\,f(\pi)\,]$ is sLR-increasing with $s(\theta) = u(w_0+\theta) - u(w_0+\theta-c)$, so

$$\frac{\mathbb{E}[\,u(w+\tilde\theta-c)\,|\,\pi\,] - \mathbb{E}[\,U_{k-1}(w-c,\Pi(\pi,\tilde x))\,|\,f(\pi)\,]}{\mathbb{E}[\,u(w_0+\tilde\theta)\,|\,\pi\,] - \mathbb{E}[\,u(w_0+\tilde\theta-c)\,|\,\pi\,]}$$

is LR-increasing. Dividing the denominator by $e^{-(w_0-w)/R}$, we get that

$$\frac{\mathbb{E}[\,u(w+\tilde\theta-c)\,|\,\pi\,] - \mathbb{E}[\,U_{k-1}(w-c,\Pi(\pi,\tilde x))\,|\,f(\pi)\,]}{\mathbb{E}[\,u(w+\tilde\theta)\,|\,\pi\,] - \mathbb{E}[\,u(w+\tilde\theta-c)\,|\,\pi\,]}$$

is LR-increasing. With an exponential utility function, this ratio is equal to

$$\frac{e^{\frac{c}{R}}\,\mathbb{E}[\,u(w+\tilde\theta)\,|\,\pi\,] - \mathbb{E}[\,U_{k-1}(w-c,\Pi(\pi,\tilde x))\,|\,f(\pi)\,]}{\mathbb{E}[\,u(w+\tilde\theta)\,|\,\pi\,]\,(1-e^{\frac{c}{R}})} \ .$$

Because $(1-e^{\frac{c}{R}}) < 0$, we then have that

$$\frac{\mathbb{E}[\,U_{k-1}(w-c,\Pi(\pi,\tilde x))\,|\,f(\pi)\,]}{\mathbb{E}[\,u(w+\tilde\theta)\,|\,\pi\,]}$$

is LR-increasing. Then

$$\mathrm{CE}(\,\mathbb{E}[\,u(w+\tilde{\theta})\,|\,\pi\,]) - \mathrm{CE}(\,\mathbb{E}[\,U_{k-1}(w-c,\Pi(\pi,\tilde{x}))\,|\,f(\pi)\,]) = R\ln\left(\frac{\mathbb{E}[\,U_{k-1}(w-c,\Pi(\pi,\tilde{x}))\,|\,f(\pi)\,]}{\mathbb{E}[\,u(w+\tilde{\theta})\,|\,\pi\,]}\right)$$

is LR-increasing because $\ln(u)$ is an increasing function. □

Let $V_k(\pi)$ be the risk-neutral value function, i.e., given by taking $u(w) = w$. In this case, the value function is independent of wealth and can be written recursively as

$$\begin{aligned} V_0(\pi) &= 0, \\ V_k(\pi) &= \max\begin{cases} \mathbb{E}[\,\tilde{\theta}\,|\,\pi\,] & \text{(adopt)} \\ 0 & \text{(reject)} \\ -c + \mathbb{E}[\,V_{k-1}(\Pi(\pi,\tilde{x}))\,|\,f(\pi)\,] & \text{(wait)} \end{cases} . \end{aligned} \tag{B2}$$

The following proposition implies Proposition 8.2 in the text.

Proposition B.1. *For a risk-averse DM with an exponential utility function and a signal process that satisfies the MLR property, we have*

$$\mathrm{CE}(\,\mathbb{E}[\,u(w+\tilde{\theta})\,|\,\pi\,]) - \mathrm{CE}(U_k(w,\pi)) \le \mathbb{E}[\,\tilde{\theta}\,|\,\pi\,] - V_k(\pi)\ . \tag{B3}$$

Proof. We have (as in the proof above)

$$\begin{aligned} &\mathrm{CE}(\,\mathbb{E}[\,u(w+\tilde{\theta})\,|\,\pi\,]) - \mathrm{CE}(U_k(w,\pi)) \\ &= \min\begin{cases} 0 & \text{(adopt)} \\ \mathrm{CE}(\,\mathbb{E}[\,u(w+\tilde{\theta})\,|\,\pi\,]) - \mathrm{CE}(\,\mathbb{E}[\,U_{k-1}(w-c,\Pi(\pi,\tilde{x}))\,|\,f(\pi)\,]) & \text{(wait)} \\ \mathrm{CE}(\,\mathbb{E}[\,u(w+\tilde{\theta})\,|\,\pi\,]) - w & \text{(reject)} \end{cases} \end{aligned} \tag{B4}$$

and

$$\mathbb{E}[\,\tilde{\theta}\,|\,\pi\,] - V_k(\pi) = \min\begin{cases} 0 & \text{(adopt)} \\ \mathbb{E}[\,\tilde{\theta}\,|\,\pi\,] - (-c + \mathbb{E}[\,V_{k-1}(\Pi(\pi,\tilde{x}))\,|\,f(\pi)\,]) & \text{(wait)} \\ \mathbb{E}[\,\tilde{\theta}\,|\,\pi\,] & \text{(reject)} \end{cases} . \tag{B5}$$

For both the certainty equivalent difference (B4) and expected value difference (B5), the terminal cases ($k = 0$) reduce to the reject cases.

We will show that (B3) holds using an induction argument. In the terminal case, we want to show that

$$\mathrm{CE}(\,\mathbb{E}[\,u(w+\tilde{\theta})\,|\,\pi\,]) - w \le \mathbb{E}[\,\tilde{\theta}\,|\,\pi\,]\ .$$

This holds because the certainty equivalent for a risk-averse utility function is less than the expected value.

For the induction hypothesis, assume, for any w and π,

$$\mathrm{CE}(\,\mathbb{E}[\,u(w+\tilde{\theta})\,|\,\pi\,]) - \mathrm{CE}(U_{k-1}(w,\pi)) \le \mathbb{E}[\,\tilde{\theta}\,|\,\pi\,] - V_{k-1}(\pi)\ .$$

We will show that each component of (B4) is less than the corresponding component of (B5). This is trivially true for the adopt option. For the reject options, this follows from the fact that the certainty equivalent is less than the expected value, as in the terminal case. So we need to study the wait case and show that

$$\begin{aligned} &\mathrm{CE}(\,\mathbb{E}[\,u(w+\tilde{\theta})\,|\,\pi\,]) - \mathrm{CE}(\,\mathbb{E}[\,U_{k-1}(w-c,\Pi(\pi,\tilde{x}))\,|\,f(\pi)\,]) \\ &\qquad \le \mathbb{E}[\,\tilde{\theta}\,|\,\pi\,] - (-c + \mathbb{E}[\,V_{k-1}(\Pi(\pi,\tilde{x}))\,|\,f(\pi)\,])\ . \end{aligned} \tag{B6}$$

Using the Δ-property for the exponential utility, we can subtract c from both sides of (B6) and (B6) is equivalent to

$$\mathrm{CE}(\,\mathbb{E}[\,u(w+\tilde{\theta}-c)\,|\,\pi\,]) - \mathrm{CE}(\,\mathbb{E}[\,U_{k-1}(w-c,\Pi(\pi,\tilde{x}))\,|\,f(\pi)\,]) \le \mathbb{E}[\,\tilde{\theta}\,|\,\pi\,] - \mathbb{E}[\,V_{k-1}(\Pi(\pi,\tilde{x}))\,|\,f(\pi)\,]\ .$$

Since taking expectations over the posteriors is equivalent to taking expectations with the prior, this (and (B6)) is equivalent to:

$$\begin{aligned}\mathrm{CE}(\,\mathbb{E}[\,\mathbb{E}[\,u(w+\tilde{\theta}-c)\,|\,\Pi(\pi,\tilde{x})\,]\,|\,f(\pi)\,]) - \mathrm{CE}(\,\mathbb{E}[\,U_{k-1}(w-c,\Pi(\pi,\tilde{x}))\,|\,f(\pi)\,]) \\ \leq \mathbb{E}[\,\mathbb{E}[\,\tilde{\theta}\,|\,\Pi(\pi,\tilde{x})\,]\,|\,f(\pi)\,] - \mathbb{E}[\,V_{k-1}(\Pi(\pi,\tilde{x}))\,|\,f(\pi)\,]\ .\end{aligned} \tag{B7}$$

Now consider the gambles involved on the left side of (B7). Let

$$a(x) = \mathrm{CE}(\,\mathbb{E}[\,u(w+\tilde{\theta}-c)\,|\,\Pi(\pi,x)\,])$$

and

$$o(x) = \mathrm{CE}(U_{k-1}(w+\tilde{\theta}-c,\Pi(\pi,x)))\ .$$

These are the certainty equivalents for adopting ($a(x)$) and following the optimal strategy ($o(x)$) conditioned on observing the signal x. The difference in certainty equivalents on the left side of (B7) can then be rewritten as:

$$\delta := \mathrm{CE}(\,\mathbb{E}[\,u(a(\tilde{x}))\,|\,f(\pi)\,]) - \mathrm{CE}(\,\mathbb{E}[\,u(o(\tilde{x}))\,|\,f(\pi)\,])\ . \tag{B8}$$

Given a signal x, because $o(x)$ follows an optimal strategy whereas $a(x)$ assumes adoption, we know that $a(x) \leq o(x)$ for each x. Thus the gamble $a(\tilde{x})$ (with random signal) is first-order stochastically dominated by $o(\tilde{x})$ and the certainty equivalent difference δ defined in (B8) must satisfy $\delta \leq 0$. Using the Δ-property of the exponential utility, we then have

$$\mathrm{CE}(\,\mathbb{E}[\,u(a(\tilde{x})-\delta)\,|\,f(\pi)\,]) - \mathrm{CE}(\,\mathbb{E}[\,u(o(\tilde{x}))\,|\,f(\pi)\,]) = 0\ , \tag{B9}$$

so the risk-averse DM is indifferent between the gambles $a(\tilde{x})-\delta$ and $o(\tilde{x})$.

From Lemma B.1, we know that the difference $a(x)-o(x)$ is decreasing in x. Thus the cumulative distribution functions for $a(\tilde{x})-\delta$ and $o(\tilde{x})$, call them $F_{a-\delta}(x)$ and $F_o(x)$, cross at most once. Given that the risk-averse DM is indifferent between these two gambles, the cumulative distributions for two gambles must cross exactly once. Furthermore, since $\delta \leq 0$, we know that $a(\tilde{x})-\delta$ is "more prone to low outcomes" than $o(\tilde{x})$, i.e., $F_o(x) - F_{a-\delta}(x)$ is first negative then turns positive (Hammond (1974)).[6] Then, from Hammond (1974), we know that a risk-neutral decision maker would prefer $a(\tilde{x})-\delta$ to $o(\tilde{x})$, so

$$0 = \mathrm{CE}(\,\mathbb{E}[\,u(a(\tilde{x}))-\delta)\,|\,f(\pi)\,]) - \mathrm{CE}(\,\mathbb{E}[\,u(o(\tilde{x}))\,|\,f(\pi)\,]) \leq \mathbb{E}[\,a(\tilde{x})-\delta\,|\,f(\pi)\,] - \mathbb{E}[\,o(\tilde{x})\,|\,f(\pi)\,]\ .$$

Using the Δ-property again, we have

$$\mathrm{CE}(\,\mathbb{E}[\,u(a(\tilde{x}))\,|\,f(\pi)\,]) - \mathrm{CE}(\,\mathbb{E}[\,u(o(\tilde{x}))\,|\,f(\pi)\,]) \leq \mathbb{E}[\,a(\tilde{x})-o(\tilde{x})\,|\,f(\pi)\,] \tag{B10}$$

Finally, from the induction hypothesis, for any signal x, we have

$$\begin{aligned}a(x)-o(x) &= \mathrm{CE}(\,\mathbb{E}[\,u(w+\tilde{\theta}-c)\,|\,\Pi(\pi,x)\,]) - \mathrm{CE}(U_{k-1}(w+\tilde{\theta}-c,\Pi(\pi,x))) \\ &\leq \mathbb{E}[\,\tilde{\theta}\,|\,\Pi(\pi,x)\,] - V_{k-1}(\Pi(\pi,x))\ .\end{aligned} \tag{B11}$$

Using this and (B10), we then have

$$\begin{aligned}\mathrm{CE}(\,\mathbb{E}[\,\mathbb{E}[\,u(w+\tilde{\theta}-c)\,|\,\Pi(\pi,\tilde{x})\,]\,|\,f(\pi)\,]) &- \mathrm{CE}(\,\mathbb{E}[\,U_{k-1}(w-c,\Pi(\pi,\tilde{x}))\,|\,f(\pi)\,]) \\ &= \mathrm{CE}(\,\mathbb{E}[\,u(a(\tilde{x}))\,|\,f(\pi)\,]) - \mathrm{CE}(\,\mathbb{E}[\,u(o(\tilde{x}))\,|\,f(\pi)\,]) \\ &\leq \mathbb{E}[\,a(\tilde{x})-o(\tilde{x})\,|\,f(\pi)\,] \\ &\leq \mathbb{E}[\,\mathbb{E}[\,\tilde{\theta}\,|\,\Pi(\pi,\tilde{x})\,]\,|\,f(\pi)\,] - \mathbb{E}[\,V_{k-1}(\Pi(\pi,\tilde{x}))\,|\,f(\pi)\,]\end{aligned}$$

The first inequality follows from (B10) and the second from (B11) and taking expectations. Thus we have established (B7), thereby completing the proof. ☐

[6] Hammond, III, J. S. 1974. Simplifying the choice between uncertain prospects where preference is nonlinear. *Management Sci.* **20**(7), 1047–1072.

One might speculate that the result of Proposition 8.2 might apply to two exponential utility functions, that is, if it is optimal to adopt with an exponential utility function with risk tolerance τ_1, then it is also optimal to adopt with an exponential utility function with risk tolerance $\tau_2 \geq \tau_1$. However, this is not true. Specifically, given the data of Table B.1 in a simple two-period problem (i.e., the DM can wait for one period) and a cost $c = 0.05$ associated with waiting, we find that for risk tolerances less than ≈0.33, it is optimal to quit immediately. For risk tolerances between ≈0.33 and ≈0.58, it is optimal to adopt immediately. For risk tolerances between ≈0.59 and ≈23, it is optimal to wait. For risk tolerances greater than ≈24, it is optimal to adopt. Thus, the optimal policies may be nonmonotonic with increasing risk tolerances, even within the exponential utility family.

Table B.1: Data for example with exponential utilities

		Likelihood	
Benefit (θ)	Priors	Negative	Positive
Low (-1)	0.05	0.10	0.90
High (70)	0.95	0.00	1.00

B.4. Proof of Proposition 9.1

Proof. To summarize terms for this proof, we define

$$g_k(\theta) = u(w_k + \delta_k\theta) - u(w_k + \delta_{k-1}\theta - \delta_k c) \ ,$$
$$a_k(\theta) = u(w_k + \delta_k\theta) - u(w_k) \ .$$

We then have

$$g'_k(\theta) = \delta_k(u'(w_k + \delta_k\theta) - \delta u'(w_k + \delta_{k-1}\theta - \delta_k c)) \ ,$$
$$a'_k(\theta) = \delta_k u'(w_k + \delta_k\theta) \ .$$

The scaling function $s(\theta)$ is defined in (12). These functions behave as follows over the following intervals.

	$\theta \leq \theta_0$	$\theta_0 < \theta < 0$	$0 \leq \theta$
$g_k(\theta)$	$-$	$+$	$+$
$g'_k(\theta)$	$+$	$+/-$	$+/-$
$a_k(\theta)$	$-$	$-$	$+$
$a'_k(\theta)$	$+$	$+$	$+$
$s(\theta)$	$+$	$+$	$+$
$s'(\theta)$	0	$-$	$-$

(B12)

Most of these claims are straightforward to check based on the definitions. To see that $g'_k(\theta) \geq 0$ for $\theta \leq \theta_0$, note that we have $w_k + \delta_k\theta \leq w_k + \delta_{k-1}\theta - \delta_k c$; then $g'_k(\theta) \geq 0$ follows because risk aversion implies the marginal utility at the lower wealth level, $u'(w_k + \delta_k\theta)$, is larger than the marginal utility at the higher wealth level, $u'(w_k + \delta_{k-1}\theta - \delta_k c)$.

We want to show that with a DARA utility function, $g_k(\theta)$ is s-increasing and $a_k(\theta)$ is s-increasing given the risk tolerance bound of the proposition. We consider three cases corresponding to the columns of (B12).

Case (i): $\theta \leq \theta_0$**.** In this region, $s(\theta) = K > 0$ and $g_k(\theta)$ and $a_k(\theta)$ are both increasing and hence s-increasing in this range.

Case (ii): $0 \leq \theta$**.**[7] In this region, $s(\theta)$ is positive and decreasing and $a_k(\theta)$ is positive and increasing. Thus $a_k(\theta)/s(\theta)$ is increasing, i.e., $a_k(\theta)$ is s-increasing.

We know that $g_k(\theta)$ is positive in this region and we want to show that $g_k(\theta)/s(\theta)$ is increasing. Taking

[7] If $c = 0$, then $\theta_0 = 0$ and $g_k(\theta) = 0$. In this case, define this region as $0 < \theta$ and include $\theta = 0$ in case (i) above.

the derivative and rearranging, this is true if

$$\frac{g_k'(\theta)}{g_k(\theta)} \geq \frac{s'(\theta)}{s(\theta)} \ . \tag{B13}$$

$g_k(\theta)$ may be increasing or decreasing in this region; $s(\theta)$ is positive and decreasing. If $g_k(\theta)$ is increasing,

$$\frac{g_k'(\theta)}{g_k(\theta)} \geq 0 \geq \frac{s'(\theta)}{s(\theta)} = \min\left\{0,\ \min_\kappa \frac{g_\kappa'(\theta)}{g_\kappa(\theta)}\right\} \ . \tag{B14}$$

and (B13) holds. If $g_k(\theta)$ is decreasing, then by construction of $s(\theta)$ in equations (11) and (12), we have

$$\frac{s'(\theta)}{s(\theta)} = \min\left\{0,\ \min_\kappa \frac{g_\kappa'(\theta)}{g_\kappa(\theta)}\right\} = \min_\kappa \frac{g_\kappa'(\theta)}{g_\kappa(\theta)}. \tag{B15}$$

The second equality above follows from the fact $g_k'(\theta)/g_k(\theta) \leq 0$ in this case. Thus (B13) holds in this case as well.

Case (iii): $\theta_0 < \theta < 0$. In this region, $s(\theta)$ and $g_k(\theta)$ behave exactly as in case (ii) and the same proof shows that $g_k(\theta)$ is s-increasing.

Now $a_k(\theta) < 0$ in this region and we want to show that $a_k(\theta)/s(\theta)$ is increasing. Taking the derivative and rearranging, we find that this is increasing if

$$\frac{s'(\theta)}{s(\theta)} \geq \frac{a_k'(\theta)}{a_k(\theta)} = \frac{\delta_k u'(w_k + \delta_k\theta)}{u(w_k + \delta_k\theta) - u(w_k)} \tag{B16}$$

for all k. This expression is analogous to (A7) in the case without discounting.

We work on the left side of (B16) first. From (11) and (12), we have

$$\begin{aligned} \frac{s'(\theta)}{s(\theta)} &= \min\left\{0,\ \min_\kappa \frac{\delta_\kappa\left(u'(w_\kappa + \delta_\kappa\theta) - \delta u'(w_\kappa + \delta_{\kappa-1}\theta - \delta_\kappa c)\right)}{u(w_\kappa + \delta_\kappa\theta) - u(w_\kappa + \delta_{\kappa-1}\theta - \delta_\kappa c)}\right\} \\ &\geq \min\left\{0,\ \min_\kappa \frac{\delta_\kappa\left(u'(w_\kappa + \delta_\kappa\theta) - u'(w_\kappa + \delta_{\kappa-1}\theta - \delta_\kappa c)\right)}{u(w_\kappa + \delta_\kappa\theta) - u(w_\kappa + \delta_{\kappa-1}\theta - \delta_\kappa c)}\right\} \ . \end{aligned} \tag{B17}$$

(The inequality follows because we are subtracting a larger number in the numerator.) Now, using the DARA assumption as in the proof without discounting, i.e., as in (A8), we have

$$\frac{\delta_\kappa\left(u'(w_\kappa + \delta_\kappa\theta) - u'(w_\kappa + \delta_{\kappa-1}\theta - \delta_\kappa c)\right)}{u(w_\kappa + \delta_\kappa\theta) - u(w_\kappa + \delta_{\kappa-1}\theta - \delta_\kappa c)} \geq -\delta_\kappa\rho_u(w_\kappa + \delta_{\kappa-1}\theta - \delta_\kappa c) \ .$$

Since the utility is assumed to be risk averse, $\rho(w) \geq 0$ for all w. The left side of (B16) satisfies

$$\frac{s'(\theta)}{s(\theta)} \geq \min_\kappa -\delta_\kappa\rho_u(w_\kappa + \delta_{\kappa-1}\theta - \delta_\kappa c) \ . \tag{B18}$$

Following the same argument as in the case without discounting (using the Taylor series approximation), the right side of (B16) satisfies

$$\frac{\delta_k u'(w_k + \delta_k\theta)}{u(w_k + \delta_k\theta) - u(w_k)} \ \leq \ \frac{1}{\theta} \ . \tag{B19}$$

Combining (B18) and (B19), we have

$$\frac{s'(\theta)}{s(\theta)} \ \geq \ \min_\kappa -\delta_\kappa\rho_u(w_\kappa + \delta_{\kappa-1}\theta - \delta_\kappa c) \ \geq \ \frac{1}{\theta} \ \geq \ \frac{\delta_k u'(w_k + \delta_k\theta)}{u(w_k + \delta_k\theta) - u(w_k)} \ . \tag{B20}$$

Thus the necessary condition (B16) holds if $\min_\kappa -\delta_\kappa \rho_u(w_\kappa + \delta_{\kappa-1}\theta - \delta_\kappa c) \geq \frac{1}{\theta}$ or, equivalently, if

$$\tau_u(w_\kappa + \delta_{\kappa-1}\theta - \delta_\kappa c) \geq -\delta_\kappa \theta \quad \text{for all } \kappa.$$

Since $u(w)$ is assumed to be DARA (i.e., $\tau_u(w)$ is increasing), we need only check this condition at the minimum possible value of θ, which in this case is the larger of the minimum value of θ or θ_0. □

B.5. The model with multiple information sources

Suppose there are L information sources with costs $c_1 \geq \ldots \geq c_L$. In this case, the value function $U_k(w,\pi)$ can be written recursively as:

$$\begin{aligned} U_0(w,\pi) &= u(w), \\ U_k(w,\pi) &= \max \begin{cases} \mathbb{E}[\, u(w+\tilde{\theta}) \,|\, \pi\,] & \text{(adopt)} \\ u(w) & \text{(reject)} \\ \mathbb{E}[\, U_{k-1}(w-c_1, \Pi_1(\pi,\tilde{x})) \,|\, f(\pi)\,] & \text{(use source 1)} \\ \vdots & \\ \mathbb{E}[\, U_{k-1}(w-c_L, \Pi_L(\pi,\tilde{x})) \,|\, f(\pi)\,] & \text{(use source L)} \end{cases} . \end{aligned}$$

Here $\Pi_\ell(\pi,x)$ denotes the posterior distribution using the likelihood functions for information source ℓ. If the utility function is increasing and the likelihood functions all satisfy the MLR property, it is easy to show that these value functions are LR-increasing using an argument like that for Proposition 4.1. These conditions also ensure that rejection policies are monotonic (as in Proposition 5.1).

To study the adoption policies, we consider the differences between the value associated with immediate adoption and the optimal value function, $G_k(w,\pi) = \mathbb{E}[\, u(w+\delta_k\tilde{\theta}) \,|\, \pi\,] - U_k(w,\pi)$. In the multiple source setting, this becomes

$$\begin{aligned} G_0(w,\pi) &= \mathbb{E}[\, u(w+\delta_0\tilde{\theta}) - u(w) \,|\, \pi\,]\ , \\ G_k(w,\pi) &= \min \begin{cases} 0 \\ \mathbb{E}[\, u(w+\delta_k\tilde{\theta}) - u(w) \,|\, \pi\,] \\ \mathbb{E}[\, u(w+\delta_k\tilde{\theta}) - u(w+\delta_{k-1}\tilde{\theta} - \delta_k c_1) \,|\, \pi\,] + \mathbb{E}[\, G_{k-1}(w-\delta_k c_1, \Pi_1(\pi,\tilde{x})) \,|\, f(\pi)\,] \\ \vdots \\ \mathbb{E}[\, u(w+\delta_k\tilde{\theta}) - u(w+\delta_{k-1}\tilde{\theta} - \delta_k c_L) \,|\, \pi\,] + \mathbb{E}[\, G_{k-1}(w-\delta_k c_L, \Pi_L(\pi,\tilde{x})) \,|\, f(\pi)\,] \end{cases} . \end{aligned}$$

The utility differences associated with information gathering from source ℓ are now

$$g_{\ell k}(w_k,\theta) = u(w_k + \delta_k\theta) - u(w_k + \delta_{k-1}\theta - \delta_k c_\ell)\ , \tag{B21}$$

where $w_k \in W_k \equiv \left[w_T - \frac{1-\delta_k}{1-\delta}c_1, w_T - \frac{1-\delta_k}{1-\delta}c_L\right]$ is the range of possible NPVs of the DM's wealth with k periods to go, after paying the costs from different information sources in all previous periods; let $\underline{w}_k = \min W_k$. We let $\theta_\ell = -c_\ell/(1-\delta)$ be the critical value for information source ℓ: the utility difference $g_{\ell k}(w_k,\theta)$ is positive if $\theta > \theta_\ell$ and negative if $\theta < \theta_\ell$.

We define the scaling functions as in the model with discounting (equations (11) and (12)), but taking minimums over a larger set of ratios, representing the different possible information sources, different wealth levels, as well as the different periods. For $\theta > \theta_1$, let

$$\phi(\theta) = \min\left\{0, \min_{\ell,k,w_k}\left\{\frac{g'_{\ell k}(w_k,\theta)}{g_{\ell k}(w_k,\theta)} : g_{\ell k}(w_k,\theta) > 0 \text{ and } w_k \in W_k\right\}\right\}\ . \tag{B22}$$

In the case of a single information source, this reduces to the definition (11) we used before: in the single source case, w_k is uniquely determined (W_k is a singleton) and $g'_{\ell k}(w,\theta) > 0$ whenever θ is greater than the critical value for that one source. With multiple information sources, we have multiple critical values and want to consider ratios only when the denominator is positive. Note that the constraint $g_{\ell k}(w_k, \theta) > 0$ is satisfied if and only if $\theta > \theta_\ell$; we will sometimes write this constraint in this form instead.

The scaling function $s(\theta)$ is then defined exactly as in the case with a single information source (12): pick some constant $K > 0$ and take $s(\theta) = K$ for $\theta \le \theta_1$ and, for $\theta > \theta_1$,

$$s(\theta) = K \exp\left(\int_{q=\theta_1}^{\theta} \phi(q)\, dq\right) , \tag{B23}$$

As before, this $s(\theta)$ is positive and (weakly) decreasing and, for $\theta > \theta_1$, $s'(\theta)/s(\theta) = \phi(\theta)$.

With this scaling function, we can then establish the analog of Proposition 9.1 with multiple information sources. Note that here, unlike Proposition 9.1, the risk tolerance bound is required for both parts of the proposition.

Proposition B.2. *Suppose the DM is risk averse and her utility function $u(w)$ is DARA. Define the scaling function $s(\theta)$ as in* (B23). *If*

$$\tau_u(\underline{w}_k + \delta_{k-1}\theta_k^* - \delta_k c_1) \ge -\theta_k^* \quad \textit{for all } k$$

where $\theta_k^* = \max\{\delta_k \underline{\theta},\ \theta_1\}$, then

(i) *$u(w_k + \delta_k\theta) - u(w_k + \delta_{k-1}\theta - \delta_k c_\ell)$, and*

(ii) *$u(w_k + \delta_k\theta) - u(w_k)$*

are s-increasing for all ℓ, k and $w_k \in W_k$.

Proof. The proof closely follows the proof for Proposition 9.1, but we have an additional case to consider. We first consider part (ii) of the Proposition.

(ii) We want to show that $a_k(w_k, \theta) = u(w_k + \delta_k\theta) - u(w_k)$ is s-increasing. For $\theta \le \theta_1$ and $\theta \ge 0$, the proofs for $a_k(w_k, \theta)$ are exactly as for $a_k(\theta)$ in Case (i) and Case (ii) in the proof of Proposition 9.1. For $\theta_1 < \theta < 0$, the proof proceeds as in Case (iii) in the proof of Proposition 9.1, except the minimums in (B17) (and thereafter) are taken over the larger set used in the definition of the scaling function (B22), rather than the set considered in (11). The result of that argument is that

$$\tau_u(w_k + \delta_{k-1}\theta - \delta_k c_\ell) \ge -\delta_k\theta \quad \text{ for all } \theta, \ell \text{ such that } \theta < \theta_\ell, \text{ all } k, \text{ and } w_k \in W_k$$

is sufficient to ensure that $a_k(w_k, \theta)$ is s-increasing. Because the utility function is assumed to be DARA (i.e., $\tau_u(w)$ is increasing), we need only check the smallest possible values of θ (which is θ_k^*) and w_k (which is $\underline{w}_k$) and the largest cost c_ℓ (which is c_1). Thus

$$\tau_u(\underline{w}_k + \delta_{k-1}\theta_k^* - \delta_k c_1) \ge -\delta_k\theta_k^* \quad \text{ for all } k$$

is sufficient to ensure that $a_k(w_k, \theta)$ is s-increasing.

(i) We want to show that $g_{\ell k}(w_k, \theta)$ is s-increasing. For $\theta \le \theta_1$, the proof for $g_{\ell k}(w_k, \theta)$ proceeds exactly like that for $g_k(\theta)$ in Case (i) in the proof of Proposition 9.1. For $\theta \ge \theta_\ell$, we know that $g_{\ell k}(w_k, \theta) > 0$ and the proof proceeds as in Cases (ii) and (iii) in the proof of Proposition 9.1, with the minimums in (B14) and (B15) are taken over the larger set used in the definition of the scaling function with multiple information sources (B22). Note that no utility assumptions (e.g., risk tolerances bounds) are required in this case.

With multiple information sources, we also have to consider the case where $\ell > 1$ and θ satisfies $\theta_1 < \theta < \theta_\ell$. Here $g_{\ell k}(w_k, \theta) < 0$ for this ℓ, but $g_{\ell' k}(w_k, \theta) > 0$ for more expensive sources ℓ'. (This case does not arise with a single information source.) To show $g_{\ell k}(w_k, \theta)$ is s-increasing in this region, we need to show

$$\frac{s'(\theta)}{s(\theta)} \ge \frac{g'_{\ell k}(w_k, \theta)}{g_{\ell k}(w_k, \theta)} . \tag{B24}$$

As in the proof for $a_k(\theta)$ in Case (iii) of Proposition 9.1, we can use the DARA assumption to show that the left side of (B24) satisfies

$$\frac{s'(\theta)}{s(\theta)} \geq -\delta_k \rho_u(w_k + \delta_{k-1}\theta - \delta_k c_\ell) \quad \text{for all } \ell \text{ such that } \theta < \theta_\ell, \text{ all } k, \text{ and } w_k \in W_k \ . \tag{B25}$$

The argument here is the same but the minimum in (B14) is taken over the larger set used in the definition of the scaling function with multiple information sources (B22).

We work on right side of (B24) next. Using a Taylor series expansion of u at $w_k + \delta_k\theta$, we can write

$$u(w_k + \delta_{k-1}\theta - \delta_k c_\ell) = u(w_k + \delta_k\theta) - \delta_k\left((1-\delta)\theta + c_\ell\right)u'(w_k + \delta_k\theta) + \frac{1}{2}\delta_k^2\left((1-\delta)\theta + c_\ell\right)^2 u''(w_0^*)$$

where $w_k + \delta_k\theta \leq w_0^* \leq w_k + \delta_{k-1}\theta - \delta_k c_\ell$ (recall that we are considering the case where $\theta < \theta_\ell$). We can then write the right side of (B24) as:

$$\begin{aligned}
\frac{g'_{\ell k}(w_k,\theta)}{g_{\ell k}(w_k,\theta)} &= \frac{\delta_k\left(u'(w_k+\delta_k\theta) - \delta u'(w_k + \delta_{k-1}\theta - \delta_k c_\ell)\right)}{u(w_k+\delta_k\theta) - u(w_k+\delta_{k-1}\theta - \delta_k c_\ell)} \\
&= \frac{\delta_k\left(u'(w_k+\delta_k\theta) - \delta u'(w_k + \delta_{k-1}\theta - \delta_k c_\ell)\right)}{\delta_k\left((1-\delta)\theta + c_\ell\right)u'(w_k+\delta_k\theta) - \frac{1}{2}\delta_k^2\left((1-\delta)\theta + c_\ell\right)^2 u''(w_0^*)} \\
&= \frac{u'(w_k+\delta_k\theta) - \delta u'(w_k + \delta_{k-1}\theta - \delta_k c_\ell)}{\left((1-\delta)\theta + c_\ell\right)u'(w_k+\delta_k\theta) - \frac{1}{2}\delta_k\left((1-\delta)\theta + c_\ell\right)^2 u''(w_0^*)} \\
&\leq \frac{u'(w_k+\delta_k\theta) - \delta u'(w_k + \delta_{k-1}\theta - \delta_k c_\ell)}{\left((1-\delta)\theta + c_\ell\right)u'(w_k+\delta_k\theta)}
\end{aligned}$$

where the inequality follows because u is assumed to be concave (thus $u''(w_0^*) \leq 0$). Rearranging, the right side of (B24) then satisfies

$$\frac{g'_{\ell k}(\theta)}{g_{\ell k}(\theta)} \leq \frac{1}{(1-\delta)\theta + c_\ell}\left(1 - \delta\frac{u'(w_k + \delta_{k-1}\theta - \delta_k c_\ell)}{u'(w_k + \delta_k\theta)}\right)$$

Because $w_k + \delta_k\theta < w_k + \delta_{k-1}\theta - \delta_k c_\ell$ in this case, we have,

$$\frac{u'(w_k + \delta_{k-1}\theta - \delta_k c_\ell)}{u'(w_k + \delta_k\theta)} < 1.$$

Since $(1-\delta)\theta + c_\ell < 0$, this implies

$$\frac{1}{(1-\delta)\theta + c_\ell}\left(1 - \delta\frac{u'(w_k + \delta_{k-1}\theta - \delta_k c_\ell)}{u'(w_k + \delta_k\theta)}\right) < \frac{(1-\delta)}{(1-\delta)\theta + c_\ell} \ \leq \ \frac{1}{\theta},$$

Thus, in this case, we have

$$\frac{g'_{\ell k}(\theta)}{g_{\ell k}(\theta)} \ \leq \ \frac{1}{\theta} \ . \tag{B26}$$

Combining (B25) and (B26), we see that the necessary condition (B24) holds if

$$\min_{\ell,\ k,w_k}\left\{ -\delta_k\rho_u(w_k + \delta_{k-1}\theta - \delta_k c_\ell) : \ell \text{ such that } \theta < \theta_\ell, w_k \in W_k \right\} \ \geq \ \frac{1}{\theta} \tag{B27}$$

or, equivalently, if

$$\tau_u(w_k + \delta_{k-1}\theta - \delta_k c_\ell) \geq -\delta_k\theta \quad \text{for all } \ell \text{ such that } \theta < \theta_\ell, \text{ all } k, \text{ and } w_k \in W_k$$

Because the utility function is assumed to be DARA (i.e., $\tau_u(w)$ is increasing), as in the part (ii) above, we need only check the smallest possible values of θ and w_k and the largest cost c_ℓ. Thus

$$\tau_u(\underline{w}_k + \delta_{k-1}\theta_k^* - \delta_k c_1) \geq -\delta_k \theta_k^* \quad \text{for all } k$$

is sufficient to ensure that(B24) holds, which implies $g_{\ell k}(w_k, \theta)$ is s-increasing in the region where θ satisfies $\theta_1 < \theta < \theta_\ell$. □

Proposition B.2 then implies that adoption policies are monotonic, using exactly the same argument as in Proposition 7.2.