Short-Term Variations and Long-Term Dynamics in Commodity Prices: Incorporating A Stochastic Growth Rate

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This note is a section from an earlier version of the paper "Short-term Variations and Long-term Dynamics in Commodity Prices," *Management Science* 46 (2000), 893-911 that develops an extension of the two-factor model described in that paper. This note uses terminology from that paper without explanation and refers to equations, figures, tables and text in that paper without full citation.

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A. Incorporating A Stochastic Growth Rate

Looking at the errors shown in Table III for the model fit to the Enron data, we see that the greatest errors are at the long-end of the futures curve. Examining the errors more closely, we find that the reason for this poor fit is that the "slope" at the long-end of the futures curve has apparently changed over time, while the model assumes this to be constant. For example, in Figure 2 we see that the slope of the model's fit to the long-term futures prices exceeds that of the actual futures price. A simple way to accommodate these changes in slope would be to use the short-term/long-term model to determine futures prices, but allow the equilibrium growth rate (μ_ξ) or its risk-adjusted counterpart (μ_ξ^*) to vary from period to period to fit then-current futures prices. This approach is easy to implement[1] and provides an improved fit to futures curves, but it is theoretically inconsistent in that it allows parameters to vary that are not treated as stochastic when valuing the futures and options on these futures. In this section, we describe an extension of the short-term/long-term model in which the growth rate for the equilibrium price (μ_ξ) is modeled as stochastic and futures and options are valued reflecting this additional source of uncertainty. This uncertainty in equilibrium growth rates may reflect uncertainty about the rate of discovery or depletion of new reserves, uncertainty about demand growth over time, and/or uncertainty about inflation. As we will see, incorporating this third factor greatly improves the model's ability to fit long-term futures prices.

A.1 The Extended Model

In this extension, we assume that the short-term deviations (χ_t) and equilibrium prices (ξ_t) follow the stochastic differential equations (1) and (2) but with the equilibrium growth rate (μ_ξ) in equation (2) being replaced by and stochastic factor μ_t that follows a stochastic process described by

$$d\mu_t \;=\; -\eta(\mu_t - \bar{\mu})\,dt \;+\; \sigma_\mu\, dz_\mu \;. \tag{A1}$$

Here dz_μ denotes increments of standard Brownian motion process that are correlated with the increments in equations (1) and (1) with pairwise correlations given by $\rho_{\chi\mu}$, and $\rho_{\xi\mu}$, respectively. Thus we assume that the long-term growth rate follows a mean-reverting process with a "natural home" or "long-term mean" equal to $\bar{\mu}$. If, for example, you believe that prices should grow with interest rates, as in Hotelling's classic model of prices for exhaustible resources, equation (A1) is equivalent to assuming that interest rates evolve as in Vasicek (1977). More generally, interest rates and equilibrium growth rates may follow distinct processes, but we might expect them to possess similar dynamics.[2]

In valuing futures contracts, we assume that the risk-neutral version of (A1) is of the form

$$d\mu_t^* \;=\; (-\eta(\mu_t - \bar{\mu}) - \lambda_\mu)\,dt \;+\; \sigma_\mu\, dz_\mu^* \;, \tag{A2}$$

with $\mu_0^* = \mu_0$ so that the risk premiums again take the form of a constant reduction in drift. Thus, the risk-neutral process for μ_t is an Ornstein-Uhlenbeck process reverting to $\bar{\mu}^* \equiv \bar{\mu} - \lambda_\mu/\eta$ rather than $\bar{\mu}$. Given this assumption, we can derive the risk-neutral joint distribution for the three-factor model following a derivation similar to that of equation (3); see the appendix. Given $\chi_o^* = \chi_0$, $\xi_o^* = \xi_0$, and $\mu_o^* = \mu_0$, we find that χ_t^*, ξ_t^*, and μ_t^* are jointly normally distributed with mean vector and covariance matrix:

$$\mathrm{E}[(\chi_t^*,\ \xi_t^*,\ \mu_t^*)] = \left[\ e^{-\kappa t}\chi_0 - (1-e^{-\kappa t})\lambda_\chi/\kappa\,,\ \xi_0 + (\bar{\mu}^* - \lambda_\xi)t + (\mu_0 - \bar{\mu}^*)\frac{(1-e^{-\eta t})}{\eta}\,,\ \mu_0 - (\mu_0 - \bar{\mu}^*)\,(1-e^{-\eta t})\ \right] \tag{A3a}$$

$$\mathrm{Cov}[(\chi_t^*,\ \xi_t^*,\ \mu_t^*)] \;=\; \begin{bmatrix} \sigma_{11}(t) & \sigma_{12}(t) & \sigma_{13}(t) \\ \sigma_{12}(t) & \sigma_{22}(t) & \sigma_{23}(t) \\ \sigma_{13}(t) & \sigma_{23}(t) & \sigma_{33}(t) \end{bmatrix} \tag{A3b}$$

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[1] Since μ_ξ^* enters equation for future prices linearly (equation 9), we could use standard Kalman filtering techniques to estimate equation μ_ξ^* as well as χ_t and ξ_t.

[2] Schwartz (1997) develops a three-factor commodity price model where the three factors are spot prices, convenience yields, and interest rates. The only impact of interest rates in that model is through discounting in the valuation of the futures and forward contracts and, in the empirical analysis, interest rates are estimated independently of spot prices and convenience yields. Here, the third factor relates directly to the futures curve and thereby provides better fits to futures prices.

where $$\sigma_{11}(t) = \sigma_\chi^2 \frac{(1-e^{-2\kappa t})}{2\kappa} ,$$

$$\sigma_{12}(t) = \rho_{\chi\xi}\sigma_\chi\sigma_\xi \frac{(1-e^{-\kappa t})}{\kappa} + \frac{\rho_{\chi\mu}\sigma_\chi\sigma_\mu}{\eta}\left(\frac{(1-e^{-\kappa t})}{\kappa} - \frac{(1-e^{-(\kappa+\eta)t})}{(\kappa+\eta)}\right) ,$$

$$\sigma_{13}(t) = \rho_{\chi\mu}\sigma_\chi\sigma_\mu \frac{(1-e^{-(\kappa+\eta)t})}{(\kappa+\eta)} ,$$

$$\sigma_{22}(t) = \sigma_\xi^2 t + \frac{\rho_{\xi\mu}\sigma_\xi\sigma_\mu}{\eta}\left(t - \frac{(1-e^{-\eta t})}{\eta}\right) + \frac{\sigma_\mu^2}{\eta^2}\left(t - 2\frac{(1-e^{-\eta t})}{\eta} + \frac{(1-e^{-2\eta t})}{2\eta}\right) ,$$

$$\sigma_{23}(t) = \rho_{\xi\mu}\sigma_\xi\sigma_\mu \frac{(1-e^{-\eta t})}{\eta} + \frac{\sigma_\mu^2}{\eta}\left(\frac{(1-e^{-\eta t})}{\eta} + \frac{(1-e^{-2\eta t})}{2\eta}\right) ,$$

$$\sigma_{33}(t) = \sigma_\mu^2 \frac{(1-e^{-2\eta t})}{2\eta} .$$

(The joint distribution for the true, as opposed to risk-neutral, stochastic process may be found by substituting zeros for the risk premiums.)

Under this risk-neutral distribution, the log of the future spot price (X_t^*) is normally distributed with:

$$\mathrm{E}[X_t^*] = e^{-\kappa t}\chi_0 - (1-e^{-\kappa t})\lambda_\chi/\kappa + \xi_0 + (\bar{\mu}-\lambda_\xi)t + (\mu_0-\bar{\mu}^*)\frac{(1-e^{-\eta t})}{\eta}$$

$$\mathrm{Var}[X_t^*] = = \sigma_{11}(t) + \sigma_{22}(t) + 2\sigma_{12}(t) .$$

Following the same analysis as in the two-factor model, we find a futures price $F_{T,0}$ satisfying

$$\begin{aligned} \ln(F_{T,0}) &= \ln(\mathrm{E}[S_t^*]) \qquad \text{(A4)} \\ &= \mathrm{E}[X_T^*] + \frac{1}{2}\mathrm{Var}[X_T^*] \\ &= e^{-\kappa T}\chi_0 + \xi_0 + (\mu_0 - \bar{\mu}^*)\frac{(1-e^{-\eta T})}{\eta} + B(T) \end{aligned}$$

where $B(T)$ depends on the time to maturity but is independent of the state variables (χ_0, ξ_0, μ_0):

$$B(T) = -(1-e^{-\kappa T})\lambda_\chi/\kappa + (\bar{\mu}^* - \lambda_\xi)T + \frac{1}{2}\big(\sigma_{11}(T) + \sigma_{22}(T) + 2\sigma_{12}(T)\big) .$$

Thus, as in the two-factor model, the log of the futures price is a linear function of the state variables. This allows us to estimate state variables over time using standard Kalman filtering techniques and estimate model parameters using maximum likelihood methods. We can also derive analytic formulas for European options following a derivation analogous to that of section 3.

As in the two-factor model, the instantaneous volatility of futures prices depends on the time to maturity but is independent of the state variables. From equation (A4), this volatility is given as

$$\sigma^2(F_{T,0}) = e^{-2\kappa T}\sigma_\chi^2 + \sigma_\xi^2 + \sigma_\mu^2\frac{(1-e^{-\eta T})^2}{\eta^2} + 2e^{-\kappa T}\rho_{\chi\xi}\sigma_\chi\sigma_\xi + 2e^{-\kappa T}\frac{(1-e^{-\eta T})}{\eta}\rho_{\chi\mu}\sigma_\chi\sigma_\mu + 2\frac{(1-e^{-\eta T})}{\eta}\rho_{\xi\mu}\sigma_\xi\sigma_\mu \,.$$

This volatility relationship is illustrated in Figure A1, using the parameter estimates from the Enron data described below. Here, as with the two-factor model, the volatility in prices for near maturity futures contracts (i.e., $T = 0$) is equal to the volatility of the sum of the short-term deviation and equilibrium levels ($\sigma^2(F_{0,0}) = \sigma_\chi^2 + \sigma_\xi^2 + 2\rho_{\chi\xi}\sigma_\chi\sigma_\xi$). As the maturity of the contract increases, the short-term deviations make less and less of a contribution to the volatility and the volatility decreases. As maturity increases more, the volatility begins to increase as the uncertainty about the equilibrium growth rate begins to play a larger role (the $\sigma_\mu^2(1-e^{-\eta T})^2/\eta^2$ term is increasing in T). As $T \to \infty$, $\sigma^2(F_{T,0})$ approaches a constant $\sigma_\xi^2 + \sigma_\mu^2/\eta^2 + 2\rho_{\xi\mu}\sigma_\xi\sigma_\mu/\eta$ (= 13.6% per year with the Enron data). Comparing this volatility curve to that of Figure 3, we see that the two- and three-factor lead to similar volatilities, with the three-factor model leading to slightly higher volatility estimates for very long-term futures contracts (13.6% vs. 11.5%). The option volatilities and observed volatilities are also shown and may be interpreted like those in Figure 3.

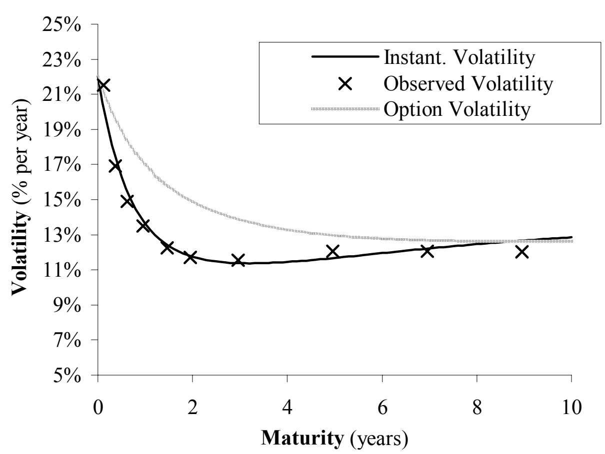

Figure A1: Volatility estimates for three-factor model.

A.2 Empirical Results

We estimate this extended model using the Kalman filtering and maximum likelihood approach described in section 5 with the Enron data. We chose not to use the futures data in this context because we felt that their relatively short maturities (up to 18 months) would not allow us to accurately identify changes in the equilibrium growth rate over time. In contrast, the Enron data includes contracts with maturities up to 9 years and changes in the expected growth rate are more transparent.

The parameter estimates for this extended model are shown in Table A1. Here we see that the model fits the observed futures prices quite well: the standard errors for the measurement equation are less than 1 percent for all but the near-term contract which has a standard deviation of error of about 2%. Overall, the likelihood function has increased from 6,182 for the two-factor model with the same data set to 7,464.[3] Examining the estimated values of new state variable (μ_t) (see Figure A2), we see that the equilibrium growth rate has changed substantially over the time horizon covered by the data set, starting around -9% and moving up to around -4% in late 1993. Because the futures prices do not directly depend on the values of μ_t (as discussed in the previous section), we do not place much confidence in the levels of the state variables μ_t presented in Figure A2. The values of the risk-adjusted growth rate ($\mu_t - \lambda_\xi$) are more reliably estimated and are also shown in Figure A2. The estimates for the other two state variables are qualitatively similar to those shown in Figure 4 and are not shown here.

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[3] We can compare these results to those obtained using Schwartz's (1997) three-factor model on this same data set (see his Table 9). There the overall likelihood function is 6,287 and the standard errors are similar to those obtained using the two-factor model.

Parameter	Description	Contract Maturity	Enron Data Estimate	Standard Error
κ	Short-term mean-reversion rate		1.26	0.03
σ_χ	Short-term volatility		14.5%	1.0%
λ_χ	Short-term risk premium		1.4%	6.0%
σ_ξ	Equilibrium volatility		13.3%	0.7%
η	Mean-reversion rate for eq. growth rate		.226	.014
$\bar{\mu}$	Mean eq. growth rate		-4.9%	8.3%
$\bar{\mu}^*$	Risk-adjusted mean eq. growth rate		-8.6	8.8%
$(\bar{\mu}^* - \lambda_\xi)$	Risk-adjusted mean eq. growth rate		0.1%	0.1%
σ_μ	Volatility in eq. growth rate		3.3%	0.2%
$\rho_{\chi\xi}$	Correlation between dev. and eq.		.267	.113
$\rho_{\chi\mu}$	Correlation between dev. and growth		-.138	.100
$\rho_{\xi\mu}$	Correlation between eq. and growth		-.524	.060
	Standard deviation(s) of error for measurement equation			
s_1	"	2 mo.	0.021	0.001
s_2	"	5 mo.	0.004	0.000
s_3	"	8 mo.	0.000	.
s_4	"	12 mo.	0.002	0.000
s_5	"	18 mo.	0.002	0.000
s_6	"	2 yrs.	0.003	0.000
s_7	"	3 yrs.	0.005	0.000
s_8	"	5 yrs.	0.006	0.001
s_9	"	7 yrs.	0.000	.
s_{10}	"	9 yrs.	0.008	0.001
NT	Number of time periods (weeks)		163	
N	Number of futures contracts		10	
	Log Likelihood		7,463.7	

Table A1: Parameter estimates for the three factor model

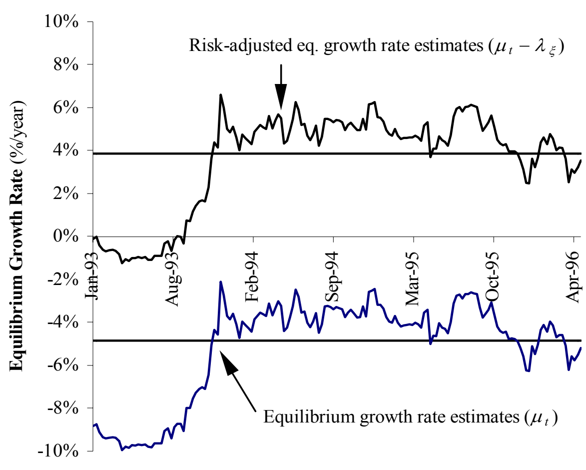

Figure A2: Estimates of the equilibrium growth rate over time

Examining the parameter estimates for three-factor in Table A1 and comparing them to the corresponding estimates for the two-factor model in Table 2, we see that those parameters that appear in both models have similar estimates. Examining the new parameters, we see that the equilibrium growth rate has a volatility (σ_μ) of approximately 3.3% per year and reverts more slowly than the short-term deviations ($\eta = .226$, corresponding to a half-life of approximately 3 years). Given the relatively short time series, we find that we cannot accurately estimate the mean growth rate ($\bar{\mu}$) or its risk-adjusted counterpart ($\bar{\mu}^* \equiv \bar{\mu} - \lambda_\mu / \eta$). We can, however, obtain a reasonably precise estimate of ($\bar{\mu}^* - \lambda_\xi$), because of the role this term plays in determining the long-term futures prices (see equation (A4)). Taking differences of these estimates, we find point estimates for the risk premiums of $\lambda_\xi = 8.7\%$ and $\lambda_\mu = 0.85\%$, with relatively large standard errors (approximately 8%). As in section 6, this reflects the fact that the risk-neutral spot price process does not depend on these risk premiums and estimates of these parameters rely on the dynamics of the implied state variable process.

In summary, this three-factor model provides much improved fits to futures prices over time. Though the formulas for valuing futures and European options are analytic, they are substantially more complicated than the corresponding formulas for the two-factor model. This additional complexity would seem worthwhile when valuing futures or options on futures with long maturities or long-lived real options.

Additional Reference

Vasicek, O. 1977, An equilibrium characterization of the term structure, *Journal of Financial Economics* 5, 177-188.

Appendix

Derivation of Equation (A3): This derivation is analogous to the derivation of equations (3). Here the transition equation is given as $\boldsymbol{x}_t = \boldsymbol{c} + \boldsymbol{Q}\,\boldsymbol{x}_{t-1} + \boldsymbol{\eta}_t$ with $\boldsymbol{x}_t \equiv [\chi_t,\ \xi_t,\ \mu_t]$, $\boldsymbol{c} \equiv [-\lambda_\chi \Delta t,\ -\lambda_\xi \Delta t,\ \eta\bar{\mu}^* \Delta t]$,

$$\boldsymbol{Q} \equiv \begin{bmatrix} \phi_1 & 0 & 0 \\ 0 & 1 & \Delta t \\ 0 & 0 & \phi_2 \end{bmatrix},$$

$\phi_1 \equiv 1-\kappa\Delta t,\ \phi_2 = 1-\eta\Delta t,\ \boldsymbol{\eta}_t$ is a vector of serially uncorrelated, normally distributed disturbances with $\mathrm{E}[\boldsymbol{\eta}_t] = \mathbf{0}$ and

$$\mathrm{Var}[\boldsymbol{\eta}_t] = \boldsymbol{W} \equiv \begin{bmatrix} \sigma_\chi^2\Delta t & \rho_{\chi\xi}\sigma_\chi\sigma_\xi\Delta t & \rho_{\chi\mu}\sigma_\chi\sigma_\mu\Delta t \\ \rho_{\chi\xi}\sigma_\chi\sigma_\xi\Delta t & \sigma_\xi^2\Delta t & \rho_{\xi\mu}\sigma_\xi\sigma_\mu\Delta t \\ \rho_{\chi\mu}\sigma_\chi\sigma_\mu\Delta t & \rho_{\xi\mu}\sigma_\xi\sigma_\mu\Delta t & \sigma_\mu^2\Delta t \end{bmatrix}.$$

Applying the same recursive procedure as before ($\boldsymbol{m}_n = \boldsymbol{Q}\,\boldsymbol{m}_{n\text{-}1}$ and with $\boldsymbol{m}_0 = \boldsymbol{x}_0 \equiv [\chi_0, \xi_0]$), the n-step ahead mean vector ($\boldsymbol{m}_n$) is given as:

$$\boldsymbol{m}_n = \begin{bmatrix} \phi_1^{\,n}\chi_0 - \lambda_\chi \sum_{i=0}^{n\text{-}1}\phi_1^i \\ \xi_0 - \lambda_\xi n\Delta t + \mu_0\Delta t\sum_{i=0}^{n\text{-}1}\phi_2^i + \eta\bar{\mu}^*\Delta t^2\sum_{i=0}^{n\text{-}2}\sum_{j=0}^{i}\phi_1^j \\ \phi_2^n\mu_0 + \eta\bar{\mu}^*\Delta t\sum_{i=0}^{n\text{-}1}\phi_2^i \end{bmatrix}.$$

Most of these expressions were encountered in the derivation of equation (3) and have similar limiting forms here. The one new expression is the double summation in the second entry. Recognizing the nested geometric series and taking the limit as $n \to \infty$ and $\Delta t = t/n \to 0$, this can be rewritten as

$$\Delta t^2\sum_{i=0}^{n\text{-}2}\sum_{j=0}^{i}\phi_2^j = \Delta t^2\sum_{i=0}^{n\text{-}2}\left(\frac{1-\phi_2^i}{1-\phi_2}\right) = \frac{\Delta t^2}{1-\phi_2}\left((n\text{-}1) + \frac{1-\phi_2^{n\text{-}2}}{1-\phi_2}\right) \to \frac{1}{\eta}\left(t + \frac{1-e^{-\eta t}}{\eta}\right).$$

Thus the mean vector approaches

$$\boldsymbol{m}_T = \begin{bmatrix} e^{-\kappa t}\chi_0 - \lambda_\chi(1-e^{-\kappa t})/\kappa \\ \xi_0 - \lambda_\xi t + \mu_0(1-e^{-\eta t})/\eta + \bar{\mu}^*(t + (1-e^{-\eta t})/\eta) \\ e^{-\eta t}\mu_0 + \bar{\mu}^*(1-e^{-\eta t}) \end{bmatrix},$$

which with some rearrangement leads to the form in equation (A3a).

To derive the covariance matrix, we proceed term by term through the matrix, beginning in each case with the terms given by applying discrete time recursion ($\boldsymbol{V}_n = \boldsymbol{Q}\,\boldsymbol{V}_{n\text{-}1}\,\boldsymbol{Q}' + \boldsymbol{W}$ with $\boldsymbol{V}_0 = \mathbf{0}$) and taking the limits as $n \to \infty$ and $\Delta t = t/n \to 0$. Let σ_{ijn} denote the entry in the ith row and jth column of $\boldsymbol{V}_n$. The first term, σ_{11n}, is similar that encountered in the derivation of equation (3b):

$$\sigma_{11n} = \sigma_\chi^2\Delta t\sum_{i=0}^{n\text{-}1}\phi_1^{2i} \;\to\; \sigma_{11}(t) = \sigma_\chi^2\frac{(1-e^{-2\kappa t})}{2\kappa}$$

The recursion for the second term yields

$$\sigma_{12n} = \rho_{\chi\xi}\sigma_\chi\sigma_\xi\Delta t\sum_{i=0}^{n\text{-}1}\phi_1^i + \rho_{\chi\mu}\sigma_\chi\sigma_\mu\Delta t^2\sum_{i=1}^{n\text{-}1}\left(\phi_1^i\sum_{j=0}^{i\text{-}1}\phi_2^j\right).$$

The first part of this expression is familiar. We can handle the second term by recognizing the nested geometric series:

$$\Delta t^2\sum_{i=1}^{n\text{-}1}\left(\phi_1^i\sum_{j=0}^{i\text{-}1}\phi_2^j\right) = \Delta t^2\sum_{i=1}^{n\text{-}1}\phi_1^i\left(\frac{1-\phi_2^{i\text{-}1}}{1-\phi_2}\right) = \frac{\phi_1\Delta t^2}{1-\phi_2}\left(\frac{1-\phi_1^{n\text{-}2}}{1-\phi_1} - \frac{1-(\phi_1\phi_2)^{n\text{-}2}}{1-\phi_1\phi_2}\right).$$

Taking the limit, we then find

$$\sigma_{12n} \to \sigma_{12}(t) = \rho_{\chi\xi}\sigma_\chi\sigma_\xi\frac{(1-e^{-\kappa t})}{\kappa} + \frac{\rho_{\chi\mu}\sigma_\chi\sigma_\mu}{\eta}\left(\frac{(1-e^{-\kappa t})}{\kappa} - \frac{(1-e^{-(\kappa+\eta)t})}{(\kappa+\eta)}\right).$$

For σ_{13n},we find a familiar form,

$$\sigma_{13n} = \rho_{\chi\mu}\sigma_\chi\sigma_\mu\Delta t\sum_{i=0}^{n-1}(\phi_1\phi_2)^i \to \sigma_{12}(t) = \rho_{\chi\mu}\sigma_\chi\sigma_\mu\frac{(1-e^{-(\kappa+\eta)t})}{(\kappa+\eta)} .$$

For σ_{22n}, the recursion yields,

$$\sigma_{22n} = \sigma_\xi^2 n\Delta t + \rho_{\xi\mu}\sigma_\xi\sigma_\mu\Delta t^2\sum_{i=0}^{n-2}\sum_{j=0}^{i}\phi_2^j + \sigma_\mu^2\Delta t^3 \sum_{i=0}^{n-2}\left(\left(\sum_{j=0}^{i}\phi_2^j\right)\left(\sum_{j=0}^{i}\phi_2^j\right)\right)$$

The limit of the first term ($\sigma_\xi^2 n\Delta t$) is straightforward. We encountered a form similar to the second term in the derivation of $\boldsymbol{m}_n$. The third term can be rewritten using

$$\Delta t^3 \sum_{i=0}^{n-2}\left(\left(\sum_{j=0}^{i}\phi_2^j\right)\left(\sum_{j=0}^{i}\phi_2^j\right)\right)= \Delta t^3\sum_{i=0}^{n-2}\left(\left(\frac{1-\phi_2^j}{1-\phi_2}\right)\left(\frac{1-\phi_2^j}{1-\phi_2}\right)\right) = \frac{\Delta t^3}{(1-\phi_2)^2}\sum_{i=0}^{n-2}(1-2\phi_2^i+\phi_2^{2i}) .$$

Taking the limit of this expression and the others in σ_{22n}, we have

$$\sigma_{22n} \to \sigma_{22}(t) = \sigma_\xi^2 t + \frac{\rho_{\xi\mu}\sigma_\xi\sigma_\mu}{\eta}\left(t - \frac{(1-e^{-\eta t})}{\eta}\right) + \frac{\sigma_\mu^2}{\eta^2}\left(t - 2\frac{(1-e^{-\eta t})}{\eta} + \frac{(1-e^{-2\eta t})}{2\eta}\right) .$$

For σ_{23n}, the recursion yields

$$\sigma_{23n} = \rho_{\xi\mu}\sigma_\xi\sigma_\mu\Delta t\sum_{i=0}^{n-1}\phi_2^j + \sigma_\mu^2\Delta t^2 \sum_{i=1}^{n-1}\left(\phi_2^i \sum_{j=0}^{i-1} \phi_2^j\right) .$$

The first term is, by now, familiar. We encountered a term similar to the second in our derivation of $\sigma_{12}(t)$, albeit with ϕ_1 in place of one of the ϕ_2. Taking the limit gives,

$$\sigma_{23n} \to \sigma_{23}(t) = \rho_{\xi\mu}\sigma_\xi\sigma_\mu\frac{(1-e^{-\eta t})}{\eta} + \frac{\sigma_\mu^2}{\eta}\left(\frac{(1-e^{-\eta t})}{\eta} + \frac{(1-e^{-2\eta t})}{2\eta}\right)$$

Finally, the expression for σ_{33} and $\sigma_{33}(t)$ are analogous to the expressions for σ_{11} and $\sigma_{11}(t)$.///