

VALUING OIL PROPERTIES: INTEGRATING OPTION PRICING AND DECISION ANALYSIS APPROACHES

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(Received May 1995; revision received March 1996; accepted September 1996)

There are two major competing procedures for evaluating risky projects where managerial flexibility plays an important role: one is decision analytic, based on stochastic dynamic programming, and the other is option pricing theory (or contingent claims analysis), based on the no-arbitrage theory of financial markets. In this paper, we show how these two approaches can be profitably integrated to evaluate oil properties. We develop and analyze a model of an oil property—either a developed property or a proven but undeveloped reserve—where production rates and oil prices both vary stochastically over time and, at any time, the decision maker may terminate production or accelerate production by drilling additional wells. The decision maker is assumed to be risk averse and can hedge price risks by trading oil futures contracts. We also describe extensions of this model that incorporate additional uncertainties and options, discuss its use in exploration decisions and in evaluating a portfolio of properties rather than a single property, and briefly describe other potential applications of this integrated methodology.

Firms routinely make decisions about whether to invest in risky projects: they must decide whether to invest some fixed amount today in exchange for an uncertain stream of future payoffs. For example, oil companies invest in exploration hoping to find valuable oil reserves. Manufacturing firms invest in new facilities and equipment hoping to streamline their future manufacturing operations and reduce production costs. Pharmaceutical firms invest in R&D hoping to develop valuable new drugs. These kinds of investments share three important characteristics: there is a great deal of uncertainty ex-ante regarding the value of the project, the firm learns more about the value of the project as it invests over time and as uncertainties are resolved, and the firm has a great deal of flexibility to adapt to this new information. For example, an oil company may abandon an exploration effort if early results prove disappointing or if oil prices fall below the levels required to justify the effort. Similarly, a manufacturing firm can shift production among facilities in different countries as exchange rates and demands vary.

Traditional discounted cash flow techniques have trouble evaluating problems with significant managerial flexibility (see, e.g., Myers 1984) and consequently, when evaluating these kinds of problems, people tend to use either decision analysis or option pricing (or contingent claims) methods. In the decision analysis approach, one values a risky project by constructing a decision tree (or dynamic program or influence diagram) that describes the sequence of decisions and uncertainties surrounding the project. The decision maker's beliefs about the project are captured by assessing subjective probabilities for the uncertainties and preferences for project cash flows are captured by using some kind of risk-adjusted discount rate or a utility function. Project values and optimal strategies are

then determined by "rolling back" the tree (i.e., through stochastic dynamic programming) and calculating expected values or utilities. This approach has its roots in statistical decision theory and was developed in the 1960s (see Howard 1966 and Raiffa 1968 for early discussions, and see Howard 1988 and Clemen 1996 for recent surveys).

The option pricing approach has its roots in the Black-Scholes-Merton methods for valuing put and call options on stock (Black and Scholes 1973, Merton 1973). The application of these methods to "real," as opposed to financial, projects dates back to Myers (1977) and Ross (1978) and was popularized by Myers (1984) and Kester (1984) (see Dixit and Pindyck 1994 for a survey of the current state-of-the-art). In this approach, rather than determining project values and optimal strategies using subjective probabilities and utilities, one seeks market-based valuations and policies that maximize these market values. In particular, one looks for a portfolio of securities and a trading strategy that exactly replicates the project's cash flows in all future times and all future states. The value of the project is then given by the current market price of this replicating portfolio. The fundamental principle underlying this approach is the "no arbitrage" principle or the so-called "law of one price": two investments with the same payoffs at all times and in all states—the project and the replicating portfolio—must have the same value.

The two approaches have complementary strengths and weaknesses. The strength of the decision analysis approach is its generality. The decision analysis paradigm provides a systematic and logical framework for making all kinds of decisions—medical treatment decisions, decisions involving environmental risks, as well as investment decisions of the kind considered here. Because of the difficulties in

Subject classifications: Decision analysis, applications: evaluating oil properties. Finance, investment: evaluating oil properties. Natural resources, energy: evaluating oil properties.

Area of review: DECISION MAKING.

formulating, assessing and solving complex models, by necessity, the decision analysis models considered in practice tend to be narrowly focused on the project at hand without considering all related decisions and uncertainties. In investment contexts, the decision analysis models rarely take into account market opportunities to hedge project risks by trading securities even though these opportunities may have an impact on the project values and, more importantly, on the optimal investment strategies.

In contrast, the option pricing approach pays careful attention to market opportunities related to the project. By explicitly constructing a portfolio of securities and a trading strategy that perfectly replicates the project cash flows, one determines project values based entirely on objective market information: all decision makers who are risk-averse (and have access to the relevant securities markets) will agree on project values and strategies regardless of their subjective beliefs and preferences. The weakness of the option pricing approach is its lack of generality. In order to determine a unique project value, one must be able to find a portfolio and trading strategy that perfectly replicates the project's cash flows. This "completeness" assumption is quite reasonable when valuing put or call options on a stock (or other derivative securities) as one can replicate the payoff of the put or call option by trading the underlying stock and a risk-free bond, but seems unrealistic for most real projects. For example, when valuing an oil property, it is reasonable to assume that price risks can be hedged by trading oil futures contracts, but it seems unreasonable to assume that reservoir-specific uncertainties, like production rate risks, can be hedged by trading securities.

Provided certain market and preference conditions are satisfied, the option pricing and decision analysis approaches can be profitably integrated. The theory underlying this integrated valuation procedure is developed in Smith and Nau (1995) and Smith (1996). Like the decision analysis approach, it is applicable in situations where some, none, or all project risks can be hedged by trading securities. Like the option pricing approach, it reduces the number of assessments required: one need not assess probabilities for market uncertainties (like oil price risks) and need not assess a full utility function. While Smith and Nau (1995) and Smith (1996) demonstrate this integrated procedure using simple examples, our primary goal in this paper is to apply this integrated procedure on a real and complex problem involving the valuation and management of oil properties. A secondary goal is to demonstrate the use of this integrated procedure in continuous-time models like those typically considered in the real options literature.

The problem of evaluating oil and gas investments is a natural place to first apply the technique. Historically, it is one of the areas where decision analysis methods were first applied (see Grayson 1960) and remains an area where decision analysis methods are heavily used (see Newendorp 1975 for an early survey and Keifer 1991 for a recent study). It is also one of the areas where option pricing

methods were first applied in nonfinancial applications (see Brennan and Schwartz 1985, Paddock et al. 1988, Lehman 1989 for early applications; see Lohrenz and Dickens 1993 and Kemna 1993 for recent reviews). In these applications, the decision analysis evaluations have typically neglected market opportunities for hedging price risks, and the option pricing evaluations have typically neglected nonfinancial risks, such as production rate uncertainty. In practice both of these risks are important and management has a great deal of flexibility to adapt as these uncertainties are resolved.

The paper is organized as follows. In Section 1, we review the assumptions and rationale underlying this integrated valuation procedure as well as the mechanics of the procedure. In Section 2, we consider a model, based on that of Olsen and Stensland (1988) and Clarke and Reed (1990), where the production capacity is fixed in that there are a fixed number of wells at the site. Production rates and oil prices vary continuously and stochastically over time and the property may be abandoned at any time. The decision maker is risk averse and can hedge price risks by trading oil futures contracts but cannot hedge production-rate risks. We use the integrated procedure to determine the value of the property (as a function of the current production rate and oil price) and the optimal policy for abandonment. We then use it to evaluate an actual property located in the Permian Basin in west Texas.

In Section 3, we extend this model and give the decision maker the option to drill additional wells to accelerate production from the reservoir, again applying the model to the Permian Basin property. As this model includes the case where no wells are currently producing, it can be used to evaluate the decision to develop a proven reserve. In Section 4, we show how the model can be extended to incorporate additional uncertainties (production costs and basis risks) and additional options (the option to temporarily suspend production), and describe how this model might be used in exploration and portfolio contexts. In Section 5, we conclude by comparing our results with more conventional procedures and discussing other potential applications of this integrated valuation procedure.

1. THE INTEGRATED VALUATION PROCEDURE

The basic idea of the integrated valuation procedure is to use option pricing methods to value risks that can be hedged by trading existing securities and decision analysis procedures to value risks that cannot be hedged by trading. In this section, we briefly describe this valuation procedure, referring the reader to Smith and Nau (1995) and Smith (1996) for more detailed and formal discussions, including proofs of the results summarized in this section.

1.1. Basic Framework

Our analysis focuses on modeling the beliefs, preferences, and decisions of a single agent, which we will refer to as the *decision maker*; this decision maker may represent an

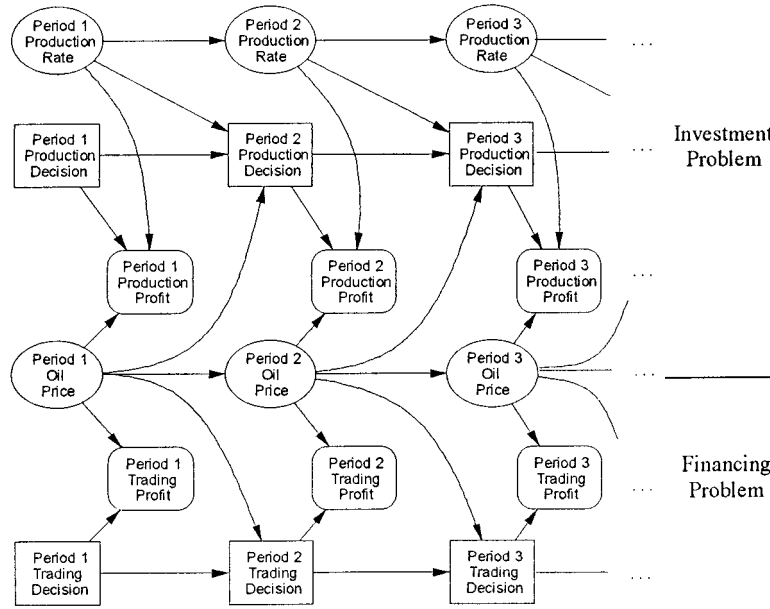


Figure 1. The grand model.

individual or, as is standard in the decision analysis literature, a firm or corporation. The valuation procedure has been developed in a discrete-time, finite-horizon framework where uncertainties are resolved and trading takes place at times $t = 0, 1, \dots, T$. The decision maker's state of information at each time t is denoted by \mathcal{F}_t , formally modeled as elements of a filtration on a suitably defined probability space. The decision maker's beliefs are captured by subjective probabilities on this space and his goal is to maximize his expected utility for consumption.

The decision maker has access to two kinds of investments: projects and securities. What distinguishes projects and securities is that projects, unlike securities, are lumpy, all-or-nothing type investments that are not traded. We imagine these projects as resulting from unique patents, land or resource rights, technical knowledge, reputation, market position, etc. that are owned (or potentially owned) by the decision maker. In contrast, we will assume that the decision maker may buy or sell as many shares of securities as desired at market prices without incurring any transactions costs. Similarly, we will assume that the decision maker may borrow and lend in any desired amount at a risk-free interest rate equal to r ($r > 0$). This is formally modeled by assuming the existence of a risk-free security whose time t price is given by $(1 + r)^t$.

The decision maker's problem is to choose (and manage) projects and trade securities so as to maximize his expected utility of consumption. Provided certain preference and market restrictions are satisfied, we can separate this *grand problem* into simpler *investment* and *financing* problems, that focus exclusively on the project at hand and exclusively on securities, respectively. We can illustrate this separation result by previewing the model developed in detail later in this paper. The grand problem is illustrated

in Figure 1. In each period, the decision maker decides how to manage the oil property (in Section 2, whether to abandon it; in Section 3, whether to drill additional wells or abandon) and how to invest in securities; specifically how many oil futures contracts and shares of the risk-free security to buy. The net cash flows to the decision maker are the sum of the cash flows generated by the oil property and those generated by trading. The oil cash flows depend on the market price for oil and the production rate at the site, both of which vary stochastically over time, as well as the production decisions. The trading cash flows depend only on the oil prices and the trading decisions.

We solve this problem by decomposing it into two parts. In the investment problem, we focus exclusively on the oil property and decisions and uncertainties related to the property (i.e., the top half of Figure 1) and use the integrated valuation procedure to determine the value of the property and the optimal project management strategy. In the financing problem, we ignore the project and focus exclusively on investments in securities (i.e., the bottom half of the Figure 1). The solution to the grand problem is given by composing the solution to these two subproblems: one manages the project according to the solution to the investment problem and invests in securities according to the solution to the financing problem. After describing the necessary market and preference restrictions in Section 1.2, we describe the integrated valuation procedure in Section 1.3.

1.2. Market and Preference Restrictions

In order to apply this integrated valuation procedure, we must place some restrictions on the form of the decision maker's utility function and on the structure of the securities market. In particular, we assume that the decision

maker's preferences for consumption $x(t)$ can be represented by a utility function that is a weighted sum of exponential utilities for individual period cash flows,

$$U(x(0), x(1), \dots, x(T)) = -\sum_{t=0}^T k(t) \exp(-x(t)/\rho(t)), \quad (1)$$

where $\rho(t) > 0$ denotes the decision maker's period- t risk tolerance and $k(t)$ describes the decision maker's time preference. The exponential form is often used in practice (see Howard 1988), and, though restrictive, it is necessary in order to be able to separate project and securities investment decisions. Moreover, this form of utility function may be seen as an approximation to a more complex "true" utility function (see Smith 1996).

We make three assumptions about the securities market. First, we assume that the securities market is *arbitrage-free* in that the decision maker cannot make profits without investing some money or taking some risks: there is no "easy money" to be made in securities trading.

Second, we assume that the securities market is *partially complete* in that the uncertainties in the model can be categorized as either *market* or *private* uncertainties. The market uncertainties are risks that can be perfectly hedged by trading securities. For example in our oil property model, future oil prices are market uncertainties: if oil prices were the only uncertainty, we could exactly replicate the property's cash flows by buying and selling oil futures contracts and shares of the risk-free security. Private uncertainties, on the other hand, are risks that cannot be perfectly hedged by trading. In the example, production rates are a private uncertainty as there are no securities whose payoffs are tied to the production rate at this particular site. We can formalize these market assumptions by defining a market state of information (or filtration) \mathcal{F}_t^m , ($\mathcal{F}_t^m \subseteq \mathcal{F}_t$) that represents the market risks resolved by time t . We assume that the decision maker can perfectly hedge these market risks.

Finally, we must assume that the market is *efficient* in that, given the current security prices, the decision maker believes that future security prices are independent of the current private information. While in our example we assume that production rates and oil prices are independent, in general contemporaneous market and private uncertainties may be dependent. For example, in agriculture or timber applications, the production rates (yields) on a farm or in a forest may be correlated with market prices as the weather may affect both simultaneously.

These market assumptions are sufficient to ensure the existence of a "risk-neutral" distribution such that the current market price $s(0)$ of a security generating a (random) dividend stream ($c(0), c(1), \dots, c(T)$) is given by

$$s(0) = \sum_{t=0}^T \frac{E^*[c(t)]}{(1+r)^t}, \quad (2)$$

where E^* denotes expectations calculated using this risk-neutral distribution and r is the risk-free rate (see Harrison and Kreps 1979). These risk-neutral probabilities will be unique if and only if the market is complete in that every project risk can be perfectly hedged by trading existing securities. If the market is complete, we can infer these risk-neutral probabilities from market prices of existing securities and then use Equation (2) to determine the market value of any project—this is the so-called "risk-neutral procedure" for solving option pricing problems. In our case, we do not assume completeness and thus cannot determine project values using this risk-neutral procedure. We can, however, determine unique risk-neutral probabilities for market events (e.g., events in \mathcal{F}_t^m), and we will use these risk-neutral probabilities in the integrated valuation procedure.

1.3. Integrated Valuation Procedure

The basic idea of the integrated valuation procedure is to use subjective beliefs and preferences to determine project values conditioned on the occurrence of a particular market state and then use the risk-neutral valuation procedure (e.g., Equation (2)) to evaluate these market-state contingent cash flows. The procedure can be described using the following modification of the standard dynamic programming "rollback" procedure. Given a project with cash flows ($c(0), c(1), \dots, c(T)$), the terminal value of the project is just the time T cash flows, e.g., $v(T) = c(T)$. At earlier times t (given state of information \mathcal{F}_t), the value of the project $v(t)$ is given by

$$v(t) = c(t) + \frac{1}{(1+r)} E^*[ECE_{t+1}[v(t+1)|\mathcal{F}_{t+1}^m, \mathcal{F}_t]|\mathcal{F}_t]. \quad (3)$$

This recursion can be broken into three steps:

STEP 1. First, calculate the *effective certainty equivalent*, $ECE_{t+1}[-]$, by taking expectations over period- t 's private uncertainties, conditioned on the outcome of that period's market uncertainties. These effective certainty equivalents are calculated using an exponential utility function with an *effective risk tolerance* equal to the sum of the decision maker's discounted future risk tolerances:

$$ECE_{t+1}[v(t+1)|\mathcal{F}_{t+1}^m, \mathcal{F}_t] \equiv -R_{t+1} \ln(E[\exp(-v(t+1)/R_{t+1})|\mathcal{F}_{t+1}^m, \mathcal{F}_t])$$

and

$$R_t \equiv \sum_{\tau=t}^T \frac{\rho_\tau}{(1+r)^{\tau-t}}.$$

STEP 2. Take expectations over the period- t market uncertainties, $E^*[-]$, using risk-neutral probabilities conditioned on the time- t state of information.

STEP 3. Discount at the risk-free rate and add in the period cash flows.

Intuitively, the first step of the procedure reduces a problem with incomplete markets to an equivalent one with complete markets; the summing of future risk tolerances reflects the decision maker's ability to spread a risk taken in one period across all future periods. The second and third steps implement the standard risk-neutral valuation procedure (e.g., Equation (2)) for complete markets. One can incorporate decision variables in this recursion in the same way as the standard dynamic programming procedure: in each period t , the decision maker selects alternatives to maximize $v(t)$. While this integrated valuation procedure has been developed in discrete time with a finite horizon, we consider its limiting behavior in Section 2 in order to evaluate the continuous-time infinite-horizon model developed there.

In evaluating projects using this integrated valuation procedure, we do not need the decision maker's probabilities for the market uncertainties or his time preferences $k(t)$. This independence is a reflection of the borrowing, lending and trading decisions lurking in the background: the decision maker borrows and lends and places bets in the securities markets so as to bring his preferences for incremental market-state-contingent cash flows into alignment with market prices. We would need these probabilities and time preferences to solve the financing problem and determine the optimal securities investments.

The justification for the values generated by this integrated valuation procedure is provided by the previously mentioned Separation Theorem: the projects and project management strategies that are optimal for the grand problem (e.g., Figure 1) are precisely those that maximize this definition of value. Since the investor is indifferent between two strategies with the same value, we can also interpret these project values as the project's *present certainty-equivalent value*: the investor is indifferent between undertaking the project and receiving $v(0)$ as a lump-sum for certain at time 0.

2. A FIXED PRODUCTION CAPACITY MODEL

To illustrate the use of the integrated valuation procedure in a real and complex example, we now turn to specific models of oil producing properties. We begin by considering a model where the number of wells producing at the site is fixed and the decision maker has the option to abandon the site at any time; production rates and oil prices are both uncertain. The problem is to determine the value of the property and the optimal policy for abandoning the property. This model is based on Olsen and Stensland (1988) and Clarke and Reed (1990), who analyzed this model using standard dynamic programming techniques without taking into account the decision maker's opportunities to hedge oil price risks or the decision maker's attitude towards risk. The model is formulated in continuous time with an infinite horizon.

2.1. The Reservoir Model

The structure of our model is illustrated in the influence diagram of Figure 1. We consider a property that consists of a single reservoir that is actively being produced, where the oil is sold at spot market prices. Though the current production rates and prices are known, the future production rates and prices are uncertain. At any time, the decision maker may continue production or may pay a one-time shutdown cost to permanently and irreversibly abandon the property. To formalize this model, we define:

- $p(t)$ \equiv the oil price per barrel at time t ,
- $q(t)$ \equiv the production rate at time t ,
- c_o \equiv fixed operating cost rate (assumed constant over time),
- c_a \equiv one-time cost of abandoning the property, and
- $1 - \gamma$ \equiv royalties, taxes and variable costs as a proportion of revenues generated.

The instantaneous net cash flow generated at time t is then given by

$$c(t) \equiv \gamma p(t) q(t) - c_o. \quad (5)$$

We will assume that the operating costs (c_o) are nonnegative, but will allow both positive or negative abandonment costs. A negative abandonment cost might arise if the scrap value for equipment at the site exceeds the other costs associated with abandonment.

Production rates and oil prices both vary stochastically over time following a random walk. Specifically, we assume that oil prices $p(t)$ and production rates $q(t)$ follow geometric Brownian motion processes, with parameters (μ_p, σ_p) and (μ_q, σ_q) representing the expected rate of change and volatility of the two processes:

$$dp(t) = \mu_p p(t) dt + \sigma_p p(t) dz_p(t), \quad (6a)$$

$$dq(t) = -\mu_q q(t) dt + \sigma_q q(t) dz_q(t), \quad (6b)$$

where dz_q and dz_p represent independent increments of standard Brownian motion processes. The current values, $p(0)$ and $q(0)$, are assumed to be known. The geometric Brownian motion model of spot prices is standard in the real options literature (see, for example Brennan and Schwartz 1985 or Dixit and Pindyck 1994). Given $p(0)$, (6a) implies that $\ln(p(t)/p(0))$ is normally distributed with mean $(\mu_p - \sigma_p^2/2)t$ and variance $\sigma_p^2 t$. As a log-normal random variable, $p(t)$ has mean $p(0) \exp(\mu_p t)$ and variance $p(0)^2 \exp(2\mu_p t) (\exp(\sigma_p^2 t) - 1)$.

The marginal distributions for production rates are similar, and in particular, the expected production at time t is given by $q(0) \exp(-\mu_q t)$; thus the stochastic process (6b) generalizes the exponential decline curve that is commonly used in petroleum engineering (see, e.g., Garb and Smith 1987). The exponential decline curve is an exact model of production rates when the production rates are proportional to the difference between reservoir and surface pressures and the reservoir pressure decreases in proportion to

the amount of oil (or gas) produced. The stochastic process (6b) generalizes this exponential model by allowing unpredictable deviations in these constants of proportionality. We will assume that $\mu_q > 0$, so production rates decline in mean over time.

Since prices and production rates enter into the cash flow Equation (5) as a revenue rate, we can greatly simplify our model by combining the stochastic processes for prices and production rates into a single process for revenue. Let $x(t) \equiv p(t)q(t)$ denote the revenue rate. Then $x(t)$ follows a stochastic process described by

$$dx(t) = \mu_x x(t) dt + \sigma_x x(t) dz_x(t),$$

where $\mu_x \equiv (\mu_p - \mu_q)$, $\sigma_x^2 \equiv \sigma_p^2 + \sigma_q^2$ and $dz_x(t)$ denotes increments of a standard Brownian motion process. We can then write the cash flows (Equation (5)) as a function of revenue rate as $c(x) \equiv \gamma x - c_o$, and, as we will see, we can describe the value of the property and the abandonment policy as a univariate function of the revenue rate, rather than a bivariate function of the production rate and oil price.

2.2. Securities

We will assume that there are two securities available, a risk-free security and an oil futures contract. The risk-free security is assumed to have a time- t price of e^{-rt} where r is the risk-free rate. The oil futures contract guarantees delivery of one barrel of oil at time τ for a specified "futures price" that is paid at delivery. To value these futures contracts, we need to make some assumptions concerning the "convenience yield" associated with oil. This convenience yield represents the value of the benefits (net of storage costs) to actually holding oil in inventory as opposed to holding a futures contract and can be thought of as being analogous to dividends paid on a stock. As is common in the real-options literature (see Brennan and Schwartz 1985 and Dixit and Pindyck 1994), we will assume that the convenience yield is proportional to the spot price of oil and given by $\kappa p(t)$. Arbitrage arguments then imply that the time- t futures price is given by $p(t)e^{(r-\kappa)(\tau-t)}$, independent of the stochastic process of the spot price (see Ross 1978). In practice, the convenience yield κ is estimated from current futures and spot prices.

These two securities are sufficient to give partially complete markets: any project whose payoffs are a deterministic function of oil prices can be replicated by trading oil futures contracts and the risk-free security. This implies the existence of a unique risk-neutral oil price process, $p^*(t)$:

$$dp^*(t) = \mu_p^* p^*(t) dt + \sigma_p p^*(t) dz_p(t),$$

where $\mu_p^* \equiv r - \kappa$ and $p^*(0) \equiv p(0)$ (see Harrison and Kreps 1979), and a combined revenue process that integrates the risk-neutral price process and subjective production rate process given by:

$$dx^*(t) = \mu_x^* x^*(t) dt + \sigma_x x^*(t) dz_x(t), \quad (7)$$

where $\mu_x^* \equiv (\mu_p^* - \mu_q)$ and $x^*(0) \equiv p(0)q(0)$.

2.3. Preferences

Finally, we will assume that the decision maker's preferences for a net cash flow stream $c(t)$, the sum of production and trading cash flows, is given by a utility function of the form of (1), generalized to continuous time and an infinite-horizon:

$$U(c(t)) = - \int_{t=0}^{\infty} k(t) \exp(-c(t)/\rho(t)) dt.$$

The decision maker's preferences are thus additive over time and exhibit constant absolute risk aversion in each period with instantaneous risk tolerance $\rho(t)$. We will assume that $\rho(t)$ is a constant ρ over time.

2.4. Valuation

Having described our model, we now use the integrated valuation procedure to determine the value of the property and the optimal policy for abandonment. Our model is formulated in continuous time with an infinite horizon, but since the valuation procedure is developed in discrete time with a finite horizon, we will evaluate the model by taking the limit of discrete-time finite-horizon approximations of the continuous-time model. Specifically, we will approximate the continuous-time price and production rate processes with discrete binomial processes and take the limit as the time interval and step size are reduced to zero. These "binomial tree" approximations are standard in the option pricing literature and were introduced by Cox et al. (1979) (see also the discussion in Dixit and Pindyck 1994).

Let h represent the length of a discrete time step, let T ($= Nh$ for some integer N) denote the horizon, and let $v_{T,h}(t, x)$ denote the approximate value of the property at time t given revenue rate $x \equiv x(t) = p(t)q(t)$. We construct our approximate model by assuming that prices, production rates, and policies remain fixed over finite time intervals of length h . We assume that the property must be abandoned at the end of the horizon (time T) and take $v_{T,h}(T, x) = -c_a$, for all x . In earlier times ($t = 0, h, 2h, \dots, T - h$), the decision maker chooses whether to abandon the property or continue production. Abandonment results in an immediate and final receipt of $-c_a$ and continuation results in a value equal to the sum of this period's cash flow and the discounted "expected certainty equivalent" of next period's value. The value at time t is then given by Equation (3) as

$$\begin{aligned} v_{T,h}(t, x) &= \max \{ -c_a, c(x)h + e^{-rh} E^* \\ &[\text{ECE}_{t+h} [\\ &v_{T,h}(t+h, x(t+h))]p(t+h), q(t)]p(t), q(t)\}. \quad (8) \end{aligned}$$

Here the effective certainty equivalent, $ECE_{t+h}[-]$, is taken with respect to the subjective production rate process, given next period's prices and this period's production rates, and the expected value, $E^*[-]$, is taken with respect to the risk-neutral price process, given current prices and production rates. As in Equation (3), the effective certainty equivalents in (8) are computed using an exponential utility function with an effective risk tolerance $R_t = \sum_{\tau=t}^T e^{-r(\tau-t)} \rho h$.

The exact solution to our problem is given by taking the limit as the horizon T recedes to infinity and the step size h is reduced to zero. In this limit, the $ECE_{t+h}[-]$ operator approaches a limiting stationary form, $ECE[-]$, where the effective certainty equivalents are calculated using an effective risk tolerance $R = \int_{\tau=0}^{\infty} \rho e^{-r\tau} dt = \rho/r$. The policies and value functions also approach limiting stationary forms that can be characterized as follows.

Proposition 1. (a) As $T \ni \infty$ and $h \ni 0$, $v_{T,h}(0, x)$ converges (pointwise) to a function $v(x)$ that is continuous, nondecreasing, and satisfies

$$rv(x) = \max \left\{ -rc_a, c(x) + \frac{1}{dt} E^*[ECE[dv(x)]] \right\}, \quad (9)$$

where $(1/dt) E^*[ECE[dv(x)]] = \lim_{h \ni 0} (1/h) E^*[ECE[v(x(t+h)) - v(x(t))|p(t+h), q(t)]|p(t), q(t)]$.

(b) If it is cheaper to operate indefinitely than it is to abandon (i.e., if $c_d/r \leq c_a$), then it is never optimal to abandon the property and we have $v(0) = -c_d/r$. Otherwise, the optimal policy is to abandon the property the first time the revenue rate drops below some threshold $x_a^* > 0$ and, at this threshold, we have $v(x_a^*) = -c_a$ and $v'(x_a^*) = 0$.

In the risk-neutral case ($\rho = \infty$), the results of the proposition follow from standard dynamic programming results for the "optimal stopping problem." In the risk-averse case, the recursion in (8) is not the standard dynamic programming recursion, but we can adapt the standard dynamic programming arguments to cover this case as well. A proof of this proposition is given in the appendix.

The recursion of part (a) of the proposition is a continuous time analog of Bellman's equation that reflects the definition of $v(x)$ as the present certainty equivalent value of the property. On the left side, we have the return the decision maker would earn if he received the lump-sum value of the property and invested it in risk-free bonds. On the right side, the first term in the maximization is the return from immediately abandoning the property, as if this money were borrowed at the risk-free rate. The second term is the expected total return from holding the property and is given by the sum of the return generated by producing at the site and the rate of change in the value of the property. The decision maker chooses a production plan to maximize return, taking into account the consequences for future values, as well as the current cash flow rates. The condition on the derivative of the value function at the abandonment is commonly referred to as the "smooth pasting" condition and can be interpreted as a

first order condition for optimality that equates the marginal values on either side of the abandonment threshold (see Dixit and Pindyck 1994 for a nice discussion of these conditions).

We can further characterize the solution by developing a differential equation that describes the value of the property when it is producing. Since the sample paths for $p(t)$ and $q(t)$ are continuous, for a small time interval h , the gamble in going from $q(t)$ to $q(t+h)$ is small and, following Pratt (1964), we can write the effective certainty equivalent as

$$\begin{aligned} ECE[v(x(t+h))|p(t+h), q(t)] \\ = E[v(x(t+h))|p(t+h), q(t)] \\ - \frac{1}{2R} \text{Var}[v(x(t+h))|p(t+h), q(t)] + o(h), \end{aligned}$$

where $R = \rho/r$ denotes the effective risk-tolerance and $o(h)$ denotes an error of order smaller than h . Then, applying standard techniques from stochastic calculus (specifically Ito's Lemma; see the appendix), we can show that the value function satisfies a differential equation of the form:

$$\begin{aligned} c(x) = rv(x) - \mu_x^* x v'(x) - \frac{1}{2} \sigma_x^2 x^2 v''(x) \\ + \frac{1}{2R} \sigma_q^2 x^2 v'(x)^2. \end{aligned} \quad (10)$$

In the case where the investor is risk-neutral (i.e., $\rho = R = \infty$), this differential equation is linear and we can write an explicit formula for the solution as

$$v(x) = \alpha_1 (x)^{\theta_1} + \alpha_2 (x)^{\theta_2} + \frac{\gamma x}{r - \mu_x^*} - \frac{c_o}{r}, \quad (11)$$

where

$$\theta_i = \frac{-(2\mu_x^* - \sigma_x^2) \pm \sqrt{(2\mu_x^* - \sigma_x^2)^2 + 8R\sigma_x^2}}{2\sigma_x^2},$$

$$\theta_1 < 0 < \theta_2,$$

and the parameters α_1 and α_2 are chosen to satisfy the boundary conditions described below. In the risk-averse case ($\rho, R > 0$), the differential Equation (10) is nonlinear, and we cannot write an explicit formula for the solution.

To completely determine the value function, we need to specify boundary conditions that depend on the policy for abandoning the property. If abandonment is not feasible or never attractive (i.e., if $c_d/r \leq c_a$), then in the risk-neutral case, the value function can be expressed analytically as

$$\begin{aligned} v(x) = E^* \left[\int_{t=0}^{\infty} e^{-rt} c(x(t)) dt \right] \\ = \int_{t=0}^{\infty} E^*[e^{-rt} c(x(t))] dt = \frac{\gamma x}{r - \mu_x^*} - \frac{c_o}{r}. \end{aligned} \quad (12)$$

So, in this case, $\alpha_1 = \alpha_2 = 0$ in Equation (11). If abandonment is feasible and at some point attractive (i.e., if $c_d/r >$

c_a), as the revenue rate increases, abandonment gets pushed further and further into the future and the value with the abandonment option approaches the value without the abandonment option. In the risk-neutral case, this then implies that $\alpha_2 = 0$ in Equation (11); using the boundary conditions from part (b) of Proposition 1, we then have

$$x_a^* = (c_o/r - c_a) \left(\frac{r - \mu_x^*}{\gamma} \right) \left(\frac{\theta_1}{\theta_1 - 1} \right), \quad \text{and} \quad (13a)$$

$$\alpha_1 = \left(\frac{-\gamma}{r - \mu_x^*} \right) \left(\frac{x_a^* (1 - \theta_1)}{\theta_1} \right). \quad (13b)$$

In the risk-averse case, we cannot write an explicit formula for the value function even in the no-abandonment case and must appeal to asymptotic results to provide the final boundary conditions. Here we find that asymptotically the value function grows in proportion to \sqrt{x} (this may be verified by substituting into Equation (10)). With the smooth pasting conditions, this leads to the following three boundary conditions for (10) in the risk-averse case with abandonment,

$$v(x_a^*) = -c_a, \quad (14a)$$

$$v'(x_a^*) = 0, \quad \text{and} \quad (14b)$$

$$\lim_{x \rightarrow \infty} v'(x) = 0. \quad (14c)$$

Using these conditions, along with the differential Equation (10), we may solve for x_a^* and $v(x)$. If abandonment is not feasible or never desirable, we substitute $v(0) = -c_o/r$ for (14a) and (14b) and solve for $v(x)$. Numerical methods for solving these kinds of "two point boundary value problems" are described in Press et al. (1986).

2.5. Numerical Results

To demonstrate the model, we apply it on an actual property located in the Permian basin in west Texas. The property consists of a single reservoir and was developed about five years ago. It is currently operating with 35 wells, producing a total of 600 barrels per day. Production is expected to decline (μ_q) at approximately 10 percent a year; the volatility in decline (σ_q) is assumed to be 3 percent per year. The operating costs are \$20 per day per well. The royalty rate is a standard one-eighth of revenues; severance and *ad valorem* taxes represent a total of 7.5 percent of revenues. Because of differences in the quality of crude produced, oil produced at this site sells for less than the standard West Texas Intermediate (WTI) price; this difference is currently \$1.50 per barrel with WTI at \$18.00 per barrel. Assuming this quality differential remains a constant fraction of the WTI price and taking into account royalties, severance and *ad valorem* taxes, we have $\gamma = 7/8 \times (1 - .075) \times (18.00 - 1.50)/18.00 = 74.192$ percent. The abandonment costs are estimated at \$10,000 per well. These values are summarized in Table I.

Our oil price assumptions are based on Gibson and Schwartz (1991), which estimates volatility ($\sigma_p = 33$ per-

Table I
Parameter Values for Example

Parameter	Symbol	Value
Costs		
Royalty and Tax Rate	$(1 - \gamma)$	25.807%
Fixed Operating Costs	c_o	\$255,500/year
Cost to Abandon	c_a	\$350,000
Production Rates		
Mean Decline Rate	μ_q	10.0% per year
Std. Dev. of Decline	σ_q	2.0% per year
Oil Prices		
Mean Growth Rate	μ_p	—
Std. Dev. of Oil Price	σ_p	33.0% per year
Markets		
Risk-Free Rate	r	0.5% per year
Convenience Yield	κ	7.7% per year

cent per year) and convenience yield ($\kappa = 7.7$ percent per year) from spot, futures, and options prices over the 1986–1988 timeframe. We do not need to make any assumptions about the mean price growth rate (μ_p) for this analysis. We take the risk-free rate to be 0.5 percent per year; this represents the average real (inflation-adjusted) return on U.S. Treasury Bills from 1925–1993 as given by Ibbotson Associates (1993). In this analysis, we use the real risk-free rate as all of the operating and abandonment costs are assumed to grow with inflation and are stated in constant 1995 dollars.

Figure 2 shows results for a risk-neutral decision maker ($R = \infty$) and for a risk-averse decision maker with an effective risk-tolerance $R = \$1.0M$. Here we see, for example, that given current conditions (prices at \$18 per barrel and production at 600 barrels a day), we have an annual revenue rate of \$3.942M per year and the property would be worth \$12.211M to a risk-neutral decision maker and \$11.508M to the risk-averse decision maker. In both cases, the value function, $v(x)$, starts as a constant $-c_a$ and remains constant until we reach the critical revenue rate x_a^* . After x_a^* , v increases, starting as a convex function of the revenue rate and then, in the risk-averse case, switches to a concave function. In the risk-neutral case, v remains convex and approaches linearity. The risk-averse values are less than the risk-neutral values; the risk premium, given by the difference between the risk-neutral and risk-averse values, increases with the revenue rate.

The critical revenue rate (x_a^*) is approximately \$260,000 per year in both the risk-neutral and risk-averse cases with the risk-neutral threshold being slightly lower than the risk-averse threshold (\$259,699 vs. \$260,037). Thus, the optimal policy calls for continuous monitoring of the revenue rate and abandoning the property the first time this rate drops below \$260,000. For example, given a WTI price of \$18 per barrel, the decision maker would not abandon the property until production fell to 39.5 barrels per day. Or, given production of 600 barrels a day, the decision maker would continue production as long as WTI prices exceed \$1.19 per barrel. In both the risk-neutral and risk-averse cases, the after-tax and royalty revenue rate at

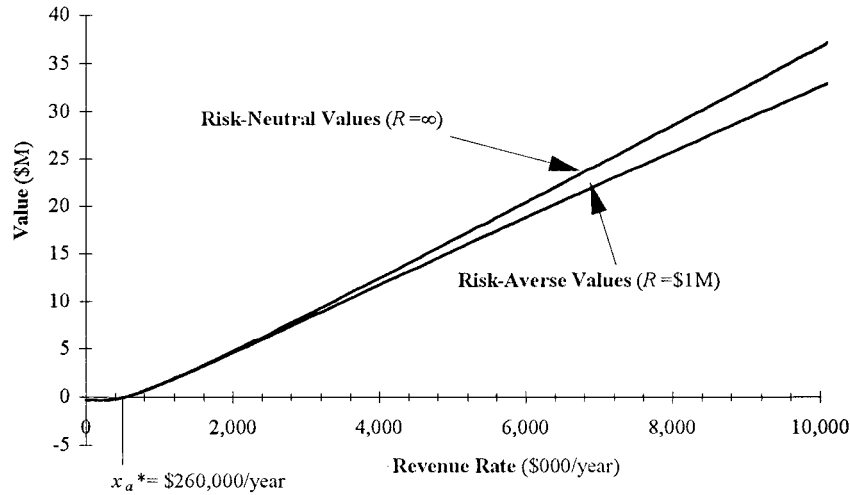


Figure 2. Value of the property as a function of current revenue rate.

the threshold, γx_a^* ($= \$192,677$ per year in the risk-neutral case) is less than the fixed operating costs ($c_o = \$255,500$ per year). Thus, it is optimal to operate at a loss, up to a point, in order to capture the potential upside associated with higher future revenue rates.

To illustrate the value of flexibility in the abandonment decision, we compare the results given by using the optimal policy to those given by two nonoptimal policies. First, we consider a policy where the abandonment time is fixed in advance. Though this kind of policy is rarely used in making the actual abandonment decision, this form is often implicitly assumed in studies that use fixed production forecasts (see, e.g., Lehman 1989). In this case, if we assume risk neutrality, we can write an explicit formula for the value of the property given a fixed time T of abandonment as in Equation (12):

$$E^* \left[\int_{t=0}^T e^{-rt} c(x(t)) dt \right] = \frac{\gamma X}{r - \mu_x^*} (1 - e^{-(r-\mu_x^*)T}) - \frac{c_o}{r} (1 - e^{-rT}).$$

One might select the abandonment time T to maximize this value, given a particular current revenue rate. For example using the parameters of Table I and a current revenue rate of \$3.942M per year, we find an optimal fixed abandonment time of 14.17 years. In this case, this fixed policy leads to a current value (for a risk-neutral decision maker) of \$11.683M as compared to a value of \$12.211M given by using the optimal policy. Here, and in general, the common approach of using a fixed abandonment time underestimates the value of a property by failing to take into account the uncertainty in future prices and production rates and the flexibility embedded in the abandonment decisions.

Another simple, nonoptimal policy that is commonly used in practice is to abandon the property when marginal revenues fall below the marginal costs of continued opera-

tion; such a policy is “myopic” in that it does not look ahead and consider the value associated with continued operation and, in particular, the option value associated with the possibility of future price increases. Examining Equation (9), the myopic policy is given by abandoning when the (after royalty and tax) marginal revenue rate, γx , is less than the marginal costs of continued operation, $c_o - rc_a$; the second term reflects interest earned by deferring the abandonment decision. We can solve for the values associated with this policy by replacing the optimal abandonment threshold x_a^* in Equations (13) and (14) with the myopic threshold $(c_o - rc_a)/\gamma$. Using the assumptions of Table I, we find that in the risk-neutral case the myopic policy calls for shutdown at a revenue rate of \$346,742 per year as compared to an optimal threshold of \$259,699 per year. Though there are substantial differences in optimal policies, the differences in values are minimal. At the current revenue rate of \$3.942M per year, the myopic policy leads to a value of \$12.162M as compared to \$12.211M for the optimal policy (again assuming risk-neutrality). As we vary the revenue rates, the values given by the myopic policy are always less than those of the optimal policy, but these differences never exceed \$52,000. Although these errors will increase with greater uncertainty in the problem (i.e., increases in either σ_q or σ_p), in general, these myopic policies lead to substantially different thresholds but approximate the optimal values quite well.¹

2.6. Sensitivity Analysis

The impacts of changes in parameter values is summarized in Table II. The table entries denote whether an increase in the indicated parameter has a positive (+) or negative (-) impact or no impact (0) on the value of the property or the abandonment threshold; the +/- entries indicate that changes in the parameter values can have either a positive or negative impact. The impacts are to be interpreted in a weak sense: a positive impact for a parameter

Table II
Sensitivity Results for Fixed Capacity Model

Parameter	Symbol	Effect of Increase in Parameter Value	
		on $v(x)$	on x_a^*
Costs			
Cost to Abandon	c_a	–	–
Fixed Operating Costs	c_o	–	+
Royalty and Tax Rate	$(1 - \gamma)$	–	+
Production Rates			
Mean Decline Rate	μ_q	–	+
Std. Dev. of Decline	σ_q	+/-	–
Oil Prices			
Mean Growth Rate	μ_p	0	0
Std. Dev. of Oil Price	σ_p	+/-	–
Markets			
Risk-Free Rate	r	+/-	+/-
Convenience Yield	κ	–	+
Preferences			
Risk Tolerance	ρ	+	–
Time Preference	$k(t)$	0	0

on, say, the abandonment threshold means that an increase in that parameter value will not decrease the threshold; it may increase the threshold or leave it unchanged. The impacts on the value function are global in that we say a parameter has a positive impact on $v(x)$ and only if it has a positive impact on $v(x)$ for all x .

Many of the impacts are easily understood. For example, increases in the costs (c_a , c_o , or $1 - \gamma$) certainly decrease cash flows and, thus, decrease the value of the property. An increase in the mean decline rate (μ_q) decreases the expected production at the site and, hence, decreases the value of the property. Similarly, an increase in convenience yield (κ) increases the value of oil in hand as compared to future oil production and decreases the value of the property. An increase in the risk tolerance (ρ) will decrease the risk premiums in the model and increase the value. In general, those parameters that have negative impacts on the value of the property increase the abandonment threshold, as one will more readily abandon a property that is worth less. Conversely those parameters with positive impacts on value will generally decrease the abandonment threshold. The one exception to this general rule is the cost to abandon (c_a): increasing c_a makes the decision maker want to delay abandonment and decreases the value of the property.

A few of the sensitivity results are perhaps surprising. The insensitivity of the value and policies with respect to changes in the growth rate in oil prices (μ_p) and the decision maker's time preferences ($k(t)$) is a reflection of the decision maker's background trading opportunities as described in the previous section. Increases in the uncertainty about future oil prices (σ_p) or production rate uncertainty (σ_q) may have either positive or negative impacts on the value of the property. On one hand, increases in uncertainty may increase the "option value" associated with the

property—it increases the "upside" associated with continued operation without affecting the "downside" risks associated with abandonment. On the other hand, if the decision maker is risk-averse, it will also increase the risk premiums associated with continued operation. For large revenue rates, the risk premium effect dominates and the values will decrease with increases in σ_p or σ_q . For rates near the threshold, the option effect dominates and values will tend to increase. At the threshold, the option effect strictly dominates and we can unambiguously conclude that the abandonment threshold decreases with increases in σ_p or σ_q .

The risk-free rate (r) has many effects in the model. It affects the mean of the risk-neutral price process ($\mu_p^* \equiv r - \kappa$) and also serves as a discount rate, both in the recursive valuation formula and in the calculation of the effective risk tolerance. Increases in the risk-free rate may have either positive or negative impacts on the value of the property. On one hand, an increase in r will further discount future revenues and increase risk premiums through its impact on the effective risk tolerance; both effects lead to a decrease in value. On the other hand, the increased discounting of future operating costs may increase the present value of the property; this is certainly the case for very small revenue rates when it is never optimal to abandon the property (i.e., $c_o/r < c_a$). The impact on the abandonment threshold is similarly ambiguous.

3. INCORPORATING PRODUCTION FLEXIBILITY

While the model of the previous sections indicates how we can apply the integrated procedure to evaluate oil and gas properties, to make the model more complete and address related theoretical and methodological issues, we consider several extensions. In this section we extend the model to consider the possibility of drilling additional wells to accelerate production at the site. For example, if oil prices increase or production rates decline slower than expected, the decision maker may find it optimal to drill additional wells. We also allow the possibility of zero wells currently producing, so the model can be used to evaluate the decision to begin development at a proven reserve.

3.1. The Reservoir Model

The basic structure of this model is the same as that of the previous section, but we must now consider the cost functions and production rates as a function of the number of wells producing and consider the possibility of drilling additional wells. To do this, we define:

- $w \equiv$ the number of wells currently producing,
- $\bar{w} \equiv$ the maximum number of wells allowed at the site,
- $q(w, t) \equiv$ the production rate given w wells producing,
- $c_o(w) \equiv$ fixed cost rate for operating w wells (assumed constant over time),
- $c_a(w) \equiv$ cost of abandoning w wells, and

$c_d(w) \equiv$ cost of drilling one additional well given w wells in place.

We will assume the oil prices $p(t)$ and royalty and tax rate $(1 - \gamma)$ do not depend on the number of wells producing. The instantaneous cash flow generated at time t with w wells is then given by

$$c(w, t) \equiv \gamma p(t) q(w, t) - c_o(w). \quad (15)$$

At any time, the decision maker may choose to: continue production; pay a one-time, lump-sum cost of $c_a(w)$ and permanently and irreversibly abandon the property; or pay a one-time, lump-sum cost of $c_d(w)$ and drill an additional well, up to the maximum number of wells allowed at the site (\bar{w}). We will assume that drilling is an instantaneous activity in that no time passes (and oil prices and base production rates do not change) between beginning and completing drilling. We also assume that the abandonment decision is an all-or-nothing decision in that the decision maker cannot abandon wells sequentially. These assumptions are for modeling convenience; if there are considerable delays in the drilling process or significant differences between the wells, one may want to consider a more refined model. The restriction to some maximum number of wells (\bar{w}) is also a modeling convenience that may hold in some situations (say in an offshore platform), but, if there is no obvious upper limit on the number of wells at the site, one may examine the values and policies with \bar{w} set to some arbitrarily large number.

In our model, the effect of drilling additional wells is to accelerate production at the site. We model this formally by introducing a base production rate $q_b(w, t)$ that represents the average per well production rate at the site; the total production is then given by $q(w, t) = wq_b(w, t)$. The base production rate then follows a geometric Brownian motion process with parameters $(-w\mu_q, \sqrt{w}\sigma_q)$:

$$dq_b(t) = -w\mu_q q_b(t) dt + \sqrt{w}\sigma_q q_b(t) dz_q(t), \quad (16)$$

where $dz_q(t)$ represents increments of a standard Brownian motion process. In this model, drilling additional wells accelerates production without affecting the total amount of oil produced: assuming production proceeds with a fixed number of wells, the production rate for w wells at time t has the same distribution as the production rate of a single well at time wt . The deterministic version of this model ($\sigma_q = 0$) is an exact model of production rates when (a) pressures are constant across the reservoir, (b) the production rate at each well is proportional to the difference between reservoir and surface pressures, and (c) reservoir pressures decline in proportion to the amount of oil (or gas) produced. Assumptions (b) and (c) are the same as those underlying the standard exponential decline model and, like (6b), the stochastic process (16) generalizes the exponential decline model by allowing unpredictable deviations in these constants of proportionality. Reservoir models with property (a) are referred to as "tank-type" models in the petroleum engineering litera-

ture and are commonly used in practice, particularly with gas or solution-gas-drive oil reservoirs (see, e.g., Steffensen 1987).

As in the fixed production capacity model, we can simplify our model by combining the stochastic processes for prices and production rates into a single process for revenue. Let $x(w, t) \equiv p(t)q_b(w, t)$ denote the base revenue rate with w wells producing. Then $x(w, t)$ follows a stochastic process described by

$$dx(w, t) = \mu_x x(w, t) dt + \sigma_x x(w, t) dz_x(t),$$

where $\mu_x \equiv (\mu_p - w\mu_q)$, $\sigma_x^2 \equiv \sigma_p^2 + w\sigma_q^2$ and $dz_x(t)$ denotes increments of a standard Brownian motion process. The cash flows (Equation (15)) can then be written as a function of the number of wells producing (w) and the base revenue rate (x) as $c(w, x) \equiv \gamma wx - c_o(w)$.

3.2. Valuation

Provided we maintain the same preference and market assumptions, the analysis of this model proceeds in the same way as the fixed-capacity model of the previous section. Assuming that we are producing with w wells, we find that the value function $v(w, x)$ describing the value of the property as a function of the number of wells operating (w) and the base revenue rate (x) satisfies a differential equation similar to (10):

$$c(w, x) = rv(w, x) - \mu_x^* x v'(w, x) - \frac{1}{2} \sigma_x^2 x^2 v''(w, x) + \frac{1}{2R} w \sigma_q^2 x^2 v'(w, x)^2, \quad (17)$$

where $\mu_x^* \equiv (\mu_p^* - w\mu_q)$ denotes the mean rate of the combined diffusion process (as in Equation (7)). In the risk-neutral case ($\rho = R = \infty$), this is a linear differential equation and we can write a closed-form solution similar to Equation (10). In the general case, (17) is a nonlinear differential equation that must be solved numerically.

As before, to completely determine the value function we must specify boundary conditions that depend on the policy for managing the well. In the case where we have already drilled the maximum number of wells allowed at the site ($w = \bar{w}$), the boundary conditions for $v(\bar{w}, x)$ are exactly analogous to the conditions for a fixed number of wells, summarized in Equations (14a)–(14c). If we have fewer than the maximum number of wells, the optimal policy is to abandon the site when the base revenue rate drops below some critical threshold $x_a^*(w)$ and drill an additional well if the base revenue rate exceeds some other threshold $x_d^*(w)$. At the abandonment threshold $x_a^*(w)$, we have boundary conditions

$$v(w, x_a^*(w)) = -c_a(w), \quad \text{and} \quad (18a)$$

$$v'(w, x_a^*(w)) = 0, \quad (18b)$$

where the first condition holds for all abandonment thresholds and the latter "smooth pasting" condition holds only if the abandonment threshold is optimal. Similarly, at the drilling threshold $x_d^*(w)$, the decision maker spends

Table III
Parameter Values for Flexible Capacity Example

Parameter	Symbol	Value
Wells		
Max. Number of Wells	\bar{w}	75
Costs		
Cost to Abandon	$c_a(w)$	\$10,000 w
Cost to Drill	$c_d(w)$	\$1,800,000 for $w = 0$; \$300,000 otherwise.
Fixed Operating Costs	$c_o(w)$	\$7,300 w per year
Royalty and Tax Rate	$(1 - \gamma)$	74.192%
Base Production Rates		
Mean Decline Rate	μ_q	0.286% per year
Std. Dev. of Decline	σ_q	0.507% per year
Oil Prices		
Mean Growth Rate	μ_p	—
Std. Dev. of Oil Price	σ_p	33.0% per year
Markets		
Risk-Free Rate	r	0.5% per year
Convenience Yield	κ	7.7% per year

$c_d(w)$ and instantaneously gets one more well. This leads to boundary conditions of the form

$$v(w, x_d^*(w)) = v(w + 1, x_d^*(w)) - c_d(w), \quad \text{and} \quad (19a)$$

$$v'(w, x_d^*(w)) = v'(w + 1, x_d^*(w)), \quad (19b)$$

where the first holds for all thresholds and the latter holds only for optimal thresholds. The value function $v(w, x)$ between the thresholds $x_a^*(w)$ and $x_d^*(w)$ is thus determined by Equations (17)–(19). For $x < x_a^*(w)$, we have $v(w, x) = -c_a(w)$. For $x > x_d^*(w)$, we have a recursive formula $v(w, x) = v(w + 1, x) - c_d(w)$. To solve this system of equations, we begin with the maximal number of wells ($w = \bar{w}$), and solve for $v(\bar{w}, x)$ and $x_a^*(\bar{w})$. We then

proceed iteratively to the cases with fewer wells, using numerical methods to find $v(w, x)$, $x_a^*(w)$ and $x_d^*(w)$ satisfying Equations (17)–(19).

3.3. Numerical Results

To illustrate the results of this extended model, we revisit the Permian Basin property and consider the original development decision where the decision maker chooses when to develop the property and how many wells to drill. We assume that each well costs \$300,000 to drill and complete, with the exception of the first well which costs \$1.8M to drill and complete; the extra \$1.5M here represents the cost of building the necessary infrastructure at the site. We take the maximum number of wells (\bar{w}) to be 75; this is far more than is optimal given current prices and production rates, and the results in this region do not vary significantly with increases in \bar{w} . The other values are the same as discussed in Section 2.5 and are summarized in Table III. The production assumptions are calculated from those assumed in the previous section where the number of wells is fixed at 35 wells. This gives parameters for the base production rate (q_b) of $\mu_q = 10/35 = .286$ percent per year and $\sigma_q = 3/\sqrt{35} = 0.507$ percent per year. As in the previous section, we present results for a risk-neutral decision maker and a risk-averse decision maker with effective risk tolerance $R = \$1M$.

The results for the example are summarized in Figure 3. The two curves represent the value functions when there are no wells producing ($v(0, x)$) in the risk-neutral and risk-averse cases. As in the previous section, we see that the two curves are close for low base revenue rates and with the difference between the two curves (representing the risk premium), increasing as the base revenue rate increases. The crosses on the curve indicate selected changes in the optimal policy. Here we find the optimal

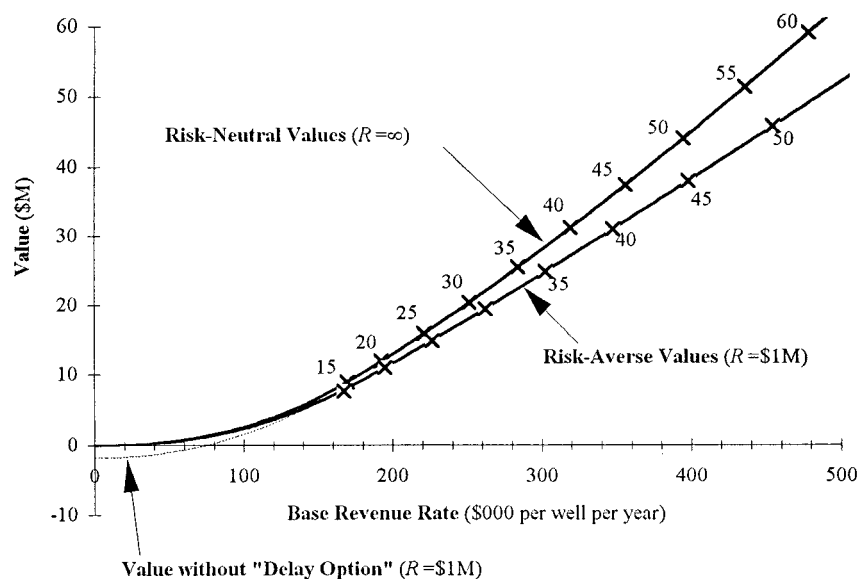


Figure 3. Value of an undeveloped property as a function of revenue rate.

policy calls for drilling no wells until the base revenue rate reaches approximately \$169,000 ($= x_d^*(0)$), in the risk-neutral case, or \$167,000, in the risk-averse case. Thus a property with initial production of 20 barrels per day per well (or 7,300 barrels per year) should not be developed until prices reach \$23.18 per barrel ($= \$169,000$ per well per year + 7,300 barrels per year); when prices reach this level, the property should be developed and 15 wells drilled.² At current prices of \$18.00 per barrel, even though it is not optimal to develop yet, the rights to develop the property still have substantial value: in the risk-neutral case, it is worth \$4.92M and, in the risk-averse case, it is worth \$4.45M. If initial production rates were 45 barrels per day per well and prices were \$18.00 barrel (for an annual base revenue rate of \$295,650 per year per well), the optimal strategy would be to develop the property and drill 35 wells in the risk-neutral case, and 33 wells in the risk-averse case.

It is interesting to note that the risk-averse decision maker would develop the property before the risk-neutral decision maker. The risk-neutral decision maker is more willing to sit on a profitable reserve, waiting for the possibility of higher oil prices, whereas the risk-averse decision maker will go ahead and develop, thus opting to receive a more certain value rather than waiting and betting on higher future prices. At higher revenue rates, the risk-averse decision maker drills fewer wells as risk-aversion decreases the overall value of the property and, hence, reduces the number of wells that can be economically justified. In both cases, the rights to the undeveloped property would never be abandoned as they may be costlessly held forever. Once developed, the thresholds for abandoning the property are slightly lower in this model than in the earlier case where there was no possibility of drilling additional wells; this difference in abandonment thresholds reflects the increased value due to the option of drilling additional wells.

To illustrate the value of the option to develop the property, Figure 3 also shows the value of the property if the decision maker had to develop it immediately. In our model, this value is given as the value with one well in place, $v(1, x)$, less the cost of drilling that first well, $c_d(0)$. For base revenue rates less than about \$80,000 per well per year, the value with immediate development is negative, whereas the value associated with waiting and pursuing the optimal development strategy is positive. Thus, though the two values converge at higher revenue rates (where immediate development is optimal), we see that properties that might be unattractive if developed immediately may have positive value if the decision maker is willing to wait and pursue the optimal development strategy. There is also a significant region (from base revenue rates ranging from \$80,000 to about \$168,000) where the optimal strategy calls for waiting even though immediate development would be profitable.

4. EXTENSIONS

In this section, we consider a number of straightforward extensions to the model developed in the previous two sections. First, we show how we can incorporate basis risks and production cost uncertainties into this model. Second, we describe how we can incorporate the ability to temporarily suspend production. We then describe how we can use the model in an exploration rather than a development context and how the model can be used to evaluate a portfolio of properties rather than a single property. We conclude with a brief discussion of some other desirable extensions that do not appear to be straightforward to incorporate into the framework of this model.

4.1. Basis Risks and Production Cost Uncertainty

Basis risks are a result of differences in quality and location between the spot market prices for the oil underlying the futures contracts and the actual oil produced at the site. For example, one might hedge price risks for a foreign crude using futures contracts assuring delivery of West Texas Intermediate in Cushing, Oklahoma. While the prices for the foreign crude will generally track prices for WTI (since they may typically be substituted for each other, albeit with transportation costs and changes in the yields of refined products), there may be random fluctuations in the relationship between the two sets of prices.

We can incorporate this kind of uncertainty into our model in a straightforward manner. Let $p(t)$ denote the spot market price of the oil underlying the futures contracts and let $p_a(t)$ denote the "actual" spot market price for the oil produced at the site. We can model the random fluctuations between the two prices by introducing a multiplicative basis factor, $b(t) = p_a(t)/p(t)$. The cash flows would then be given by

$$c(w, t) \equiv \gamma b(t) p(t) q(w, t) - c_o(w),$$

rather than Equation (15). If we assume that these basis risks are an unhedgable, private risk and assume $b(t)$ follows a geometric Brownian motion with parameters (μ_b, σ_b) , we can incorporate this additional uncertainty into the revenue rate process by taking $\mu_x \equiv (\mu_p + \mu_b - w\mu_q)$, $\sigma_x^2 \equiv \sigma_p^2 + \sigma_b^2 + w\sigma_q^2 + 2\rho_{pb}\sigma_p\sigma_b$, where ρ_{pb} represents the correlation between the price and basis risk processes. The valuation would proceed as before, leading to a differential equation of the form of Equation (17) with μ_x^* (the mean rate of the combined revenue process) replaced by $(\mu_p^* + \mu_b - w\mu_q)$ and $w\sigma_q^2$ (the total variance of the private risks) replaced by $(\sigma_b^2 + w\sigma_q^2)$. Thus, basis risks can be incorporated by changing the model parameters and affect the values and policies in the same way as production rate uncertainty.

Another possible source of uncertainty is in the variable costs of production. These costs of production may change over time, perhaps due to technical advancements in the production technology or changes in the production mix (water, oil, and various gases) at the site. This uncertainty

can be modeled in the same way as basis risks. If we introduce another stochastic multiplicative factor into the cash flow Equation (15) and make similar assumptions about the stochastic process governing these costs, we would get similar modifications to the valuation Equation (17).

4.2. Temporary Production Stoppages

In the models of Sections 2 and 3, we have given the decision maker the option to abandon a property but have not incorporated the option to temporarily halt production at the site. While this option may not always be available (perhaps due to contractual or regulatory requirements), it might be quite attractive when available, particularly in low price environments.

It is straightforward to incorporate a temporary shut-down option into the model of Section 3. The effect of temporarily shutting down is to stop production (and decline); prices would continue to fluctuate. If we let s denote a discrete state variable with $s = 1$ indicating the property is currently producing and operating at full capacity and $s = 0$ indicating the property is temporarily closed, the new cost function $c(w, x, s)$ and value function $v(w, x, s)$ satisfy equations of the form of (15) and (17) except one replaces w , representing the number of wells at the site, with sw , representing the number of wells currently operating. There may be lump-sum costs associated with halting and subsequently resuming production as well as a change (likely a reduction) in fixed operating costs (c_o) when the site is closed.

The optimal policy would then consist of specifying, for each w , thresholds at which new wells should be drilled ($x_d(w, 1)$) and at which the property should be closed or abandoned ($x_a(w, 1)$), given that the site is producing. We would also need to specify (for each w) thresholds for reopening and abandonment ($x_d(w, 0)$ and $x_a(w, 0)$), given that the property is closed. To determine these thresholds and the value function, we impose boundary conditions that are analogous to those of Equations (18) and (19).

4.3. Exploration Decisions

In our model, we have consistently assumed that the decision maker knows the current production rates and oil prices at the time he makes his production decisions. While this assumption makes sense when evaluating a producing property or a proven reserve, as we move further upstream and consider exploration as well as development activities, this assumption may no longer be reasonable. Before a reserve has been proven, there may be significant uncertainty about the production rates at the site, and the prices prevailing at the time when the exploration is completed (and the reservoir proven). In this context, our model can be used as an “endpoint” model as illustrated in the tree of Figure 4. Here the decision maker would specify probabilities describing the uncertainty about the initial production rate and use risk-neutral probabilities for prices at the time exploration is completed. We would then

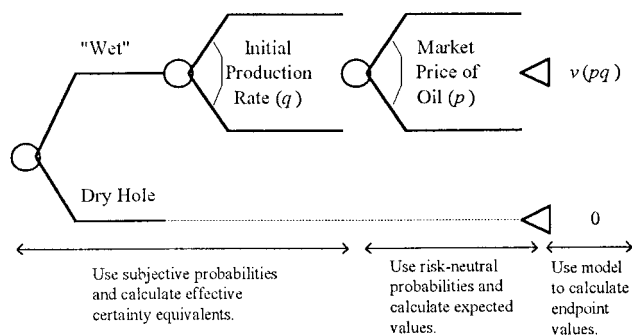


Figure 4. Exploration tree.

use our model to determine the value of the well (and policy for managing the well) in each of these initial production rate/price scenarios. In this extension, we need not make specific assumptions about the distribution of initial production rates: one could, for example, assign a significant probability to the reserve proving to be “dry hole” with no oil and then assign, say, a normal distribution to the initial production rate given that the site is not dry.

In evaluating this extended model, we would again use the integrated valuation procedure. We would discount the future values of the reservoir back to present values using the risk-free discount rate, and when “rolling back” the tree, at the oil price node, compute expected values using the risk-neutral distribution for oil prices. At the nodes representing geologic uncertainties, we would use exponential utilities and subjective probabilities to calculate effective certainty equivalents. The result is a value for the exploration program that takes into account market opportunities to hedge oil price risks, during exploration as well as development, and the decision maker’s flexibility in managing the reserve once proven.

One could incorporate uncertainty about other parameters of the “endpoint model” in a similar fashion. For example, we could include uncertainty about the mean decline rate (μ_q) or the fixed costs of operation (c_o) by including additional nodes in the tree. In doing this, we would be assuming that all uncertainty about these parameters is resolved at the time exploration is completed. One might also include exploration decisions in the tree (e.g., should they do 3D seismic studies? should they do extended well testing?) and consider a sequential process where later exploration decisions are made knowing the outcome of earlier exploration activities.

4.4. Portfolio Valuation

Finally, we consider how this model might be used to value a portfolio of properties rather than a single property. If the decision maker is risk neutral, we can use our model to value the individual properties in the portfolio and then sum to find the value of the portfolio. If the decision maker is risk averse, we need to not only value the individual properties but must also take into account the correlations among the properties. The properties are naturally

correlated through their common dependence on oil prices. If we do not take into account market opportunities for hedging price risks, this correlation would lead to substantial risk premiums in the portfolio evaluation and the value of the portfolio would be less than the sum of the values calculated for the individual properties. But if we take into account market opportunities for hedging, because this common price risk may be perfectly hedged, the portfolio value would be equal to the sum of the individual properties' values, provided there are no common nonmarket (private) risks. (This is proven in Smith 1996.)

In practice, there may be common private risks as well as common market risks. For example, prospects that lie in the same geographic region (e.g., in the same field), particularly in undeveloped regions, may have correlated production or cost uncertainties. This correlation could affect values at the portfolio level and may affect optimal development strategies. For example, the information from one reservoir may provide valuable information about the production possibilities at nearby prospects and sequential development strategies may be preferred to parallel development strategies.

4.5. Other Extensions

There are, of course, a number of other directions in which one might want to extend the models developed in this paper. One might consider alternative stochastic processes for prices (for example, various "mean-reverting" models, see e.g., Dixit and Pindyck 1994), more complex production models (for example, models that incorporate periods of flat production or more complex drive mechanisms), more complicated financial terms (for example, more complicated royalty and tax structures), or models that incorporate "learning" about parameters of the production process or costs over time. In the models we have developed and analyzed in this paper, the production and price uncertainties combine to give a single state variable corresponding to the revenue rate and the parameters of the model are assumed to be stationary; these features greatly simplify the description and computation of values and policies. Unfortunately, most of these other extensions would seem to require a significant expansion of the state space and the introduction of nonstationary parameters. While the basic valuation procedure (as described in Equation (3)) would remain the same, these extended models would be significantly more difficult to formulate and solve.

5. CONCLUSIONS

In this paper, we have demonstrated how option pricing and decision analysis techniques can be integrated to solve practical problems where the decision maker has significant managerial flexibility and may hedge some, but not all, of the risks associated with the project. Our particular example is a model for evaluating oil producing properties that takes into account price and production rate uncertainties, management's ability to accelerate or terminate

production, the decision maker's attitude toward risk, and opportunities for hedging price risks. This model could be used, for example, to value existing properties, to determine strategies for developing properties that are already owned, or to help determine bids for offshore oil and gas leases. We conclude by comparing our results with those generated by more conventional methods and describing some other potential applications of the methodology.

5.1. Comparison with Conventional Techniques

To illustrate the benefits of the valuation methodology used in this paper, it is useful to compare our results to the results given by using conventional methods with our models. While Smith and Nau (1995) provide a more general comparison, we will focus on the results given in our particular application. We will consider two different conventional "decision analysis" approaches; in the first, we attempt to determine appropriate risk premiums by adjusting the discount rate and, in the second, we use a utility function to capture time and risk preferences. We also consider the conventional option pricing approach.

Risk-adjusted Discount Rate Approach. A common approach for solving valuation problems is to select a risk-adjusted discount rate that represents the market-required rate of return for a given project. One might, for example, attempt to estimate the correlation between the project cash flows and the market as a whole (or the "beta" for the project) and estimate an appropriate risk-adjusted discount rate using, for example, the Capital Asset Pricing Model. Because of the difficulty of estimating betas for a particular project, firms often look for traded securities that are "similar in risk" to the project and use their historical returns or betas, or, alternatively use a single discount rate for all projects in their portfolio, perhaps corresponding to the firm's weighted average cost of capital (see, e.g., Brealey and Myers 1984, Ch. 9).

To illustrate the results given by this approach, we reconsider the example of Section 2 using conventional dynamic programming methods. In solving the model, we use the parameters of Table I and arrive at a solution of the form of Equations (11) and (13) except, in accordance with the conventional dynamic programming procedure, we use the true price process (with mean growth rate μ_p) rather than the risk-neutral price process (with mean growth rate $\mu_p^* = r - k$) and use the risk-adjusted discount rate rather than the risk-free rate. We take the true mean of the price process to be equal to the historical estimate (based on annual data from 1900–1995) of 0 percent, representing an expectation of zero real price growth, and consider values and policies for three different discount rates: 7, 11, and 15 percent. The original valuations of the property were performed using a corporate-wide discount rate of 11 percent, corresponding to a 15 percent nominal rate with an assumed 4 percent inflation rate.

The results are shown in Figure 5. If the current revenue rate is \$3.942M per year (as it is currently), then the value

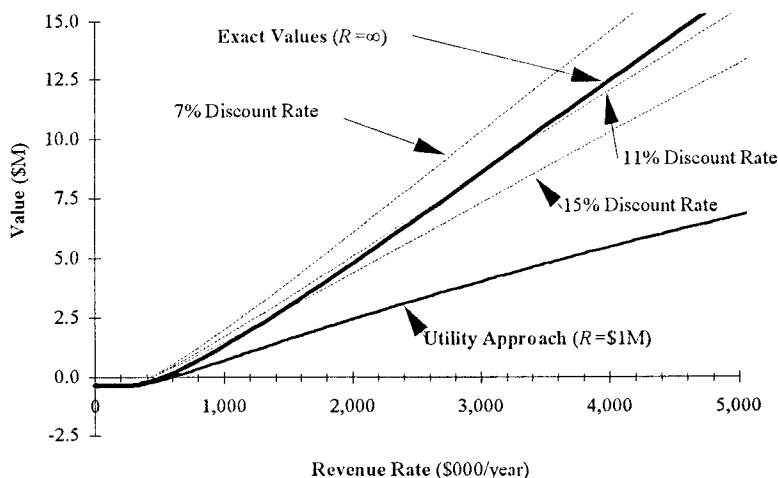


Figure 5. Property values for different discount rates.

of the property with a 15-percent discount rate is \$10.1M; with an 11-percent rate, \$11.8M; and with a 7-percent rate, \$14.3M. If we compare these results to the “correct” results given by the integrated procedure, assuming risk neutrality, we find a value of \$12.2M at the current revenue rate, and, assuming an effective risk tolerance of \$1M, a value of \$11.5M. In general, we see that the 11-percent rate originally used by the company underestimates the value of the property at high revenue rates (when the time horizons are long) and overestimates the values at lower revenue rates (when the time horizons are short). There is also disagreement about the optimal policy. If we discount at 7 percent, the property should be abandoned when revenue rates drop below \$197,000 per year; at 11 percent the threshold is \$207,000 per year; and at 15 percent the threshold is \$215,000 per year. The correct abandonment threshold, approximately \$260,000 per year, is significantly higher than that given by any of these discount rates. While we can always find a discount rate that gives the correct value for any particular revenue rate or the correct abandonment threshold, there is no single discount rate that we could use in the conventional dynamic programming approach that would generate correct values and policies for all revenue rates—one cannot take into account risk aversion and market opportunities to trade by simply adjusting the discount rate.

Utility Approach. In the decision analysis literature, it is often suggested (see, e.g., McNamee and Celona 1990) that rather than using a risk-adjusted discount rate, one should capture the time value of money using the risk-free discount rate and capture risk preferences using a utility function. In this approach, one would use the true probabilities for all uncertainties (e.g., a mean price growth rate of $\mu_p = 0$ percent rather than the risk-neutral rate $\mu_p^* = r - k = -7.2$ percent) and would assign risk premiums to all uncertainties. By failing to recognize that price risks can be managed by buying and selling futures contracts, this procedure would overstate the risks that the decision

maker would actually bear and consequently would overestimate the risk premiums for a risk-averse decision maker. In this example, the price risks are substantial and, as shown in Figure 5, the values given by the utility approach are much lower than the correct values. We can, of course, correct the problems of the utility approach by expanding the model to explicitly include market opportunities to trade, as in the “grand model” of Figure 1. Unfortunately, these grand models are quite complex and difficult to formulate and solve.

Option Pricing Approach. Finally, we could also simplify our model and apply standard option pricing methods. To use the standard option pricing methods, we must assume that we can hedge all project risks by trading marketed securities which, in this model, is equivalent to assuming that there is no uncertainty in production rates (i.e., $\sigma_q = 0$), basis risks, or costs. If we make this assumption, we could then use risk-neutral pricing techniques and find values and policies that are independent of the decision maker’s preferences. But in neglecting these unhedgable risks, we understate the uncertainty in the problem and, consequently, understate the “option value” generated by the unhedgable production rate uncertainty and, in the risk-averse case, neglect the risk premiums associated with these risks.

The integrated approach allows some of the simplifications provided by the option pricing approach without making such strong assumptions about the completeness of financial markets. Like the option pricing approach, we are able to work with small models that focus exclusively on the project at hand (i.e., we work with the models of Sections 2 and 3 rather than the grand model of Figure 1). In using risk-neutral distributions for the market uncertainties and discounting cash flows using the market’s risk-free rate, this integrated approach implicitly takes into market opportunities for trading and borrowing and, provided the necessary preference and market assumptions are satisfied, produces results identical those that would be

found if we were to formulate and solve the grand model of Figure 1.

5.2. Other Applications

Firms in the oil and gas industry are obviously interested in techniques for valuing oil and gas investments as these problems are at the heart of the “upstream” (the exploration and development) side of their business. This methodology also has many potential applications outside the oil and gas industry. The basic methodology is applicable whenever a firm (or individual) is contemplating investments with uncertain outcomes and there are market opportunities to hedge some of the risks associated with the investments. To get a sense of the range of potential applications, we briefly describe three other areas where the integrated valuation procedure could be applied.

Commodity Related Businesses. Given the detailed example of valuing an oil property, the applicability of this methodology to other commodity related businesses should be apparent. For example, one could model other natural resource assets—for example, copper mines (as in Brennan and Schwartz 1984) or forests (as in Morck et al. 1989)—and use the integrated procedure to determine the values and optimal policies for managing these assets. Similarly, one could consider the consumers of these commodities—for example, an electric utility or steel plant—and use the integrated procedure to determine, say, optimal policies for flexible generation and production systems. The integrated methodology would allow us to treat both market and nonmarket uncertainties. In these applications as in our oil property model, we can use commodity futures, swaps, and options markets to help evaluate the commodity price risks and use internal information to describe the nonmarket risks (e.g., production or demand uncertainties).

Multinational Production and Distribution. Another area of potential applications concerns balancing production and distribution among plants and markets in different countries. Prices, demands, and costs will vary over time and by country and management has the ability to adjust production levels between products and plants from period to period in response to changes in prices, demand or costs. These kinds of problems have been studied recently by Huchzermeier and Cohen (1992) and Kogut and Kulatilaka (1994). While these two models differ substantially in their details, they both assume that all uncertainty in the problem is generated by exchange rate fluctuations; given exchange rates, local costs, demands, and prices are all assumed to be known. The integrated methodology would allow the treatment of a richer set of uncertainties (e.g., cost, demand and price uncertainty), while taking into account risk preferences and opportunities for hedging exchange rate risks.

Research and Development. To examine a more difficult application, let us consider the problem of evaluating an

R&D project, for example, a pharmaceutical firm investing in research with hopes of finding a new drug. In these problems, there is a great deal of uncertainty regarding the value of the project and management has a great deal of flexibility in managing the R&D process; perhaps one in 10,000 compounds tried will eventually be a commercial success. Though many have talked about an R&D as being analogous to a call option on a stock (see, e.g., Myers 1984), we know of no serious applications of option pricing methods in this area. One reason for this lack of rigorous applications is the difficulty of finding replicating portfolios given the technical nature of the uncertainties associated with an R&D project. In contrast, decision analysis methods are commonly used to evaluate R&D projects (see, e.g., the articles in Howard and Matheson 1984).

Since the integrated procedure does not require the existence of a perfect replicating portfolio, it may allow one to use option pricing methods in evaluating R&D projects, even though in many of these applications the link between the R&D project and existing securities markets may not be immediately obvious. How can we hedge the risks associated with a pharmaceutical research project? If there are no relevant market uncertainties, then the integrated procedure essentially reduces to the standard decision analysis procedure: we would evaluate the project using private information only and discount at the risk-free rate. But in actual evaluations, practitioners often assume a positive correlation between the value of the project and the market portfolio (e.g., a positive beta) and use discount rates that are much higher than the risk-free rate. For example, Myers and Shyam-Sunder (1991) assume that the value of a new drug has a positive correlation with the market portfolio (a beta of 0.75 in their example) and study the amplification of this correlation in a sequential development process.

To capture this kind of market risk in the integrated framework, we would have to include the value of the market portfolio as a variable in the model and explicitly model the dependence between the project and the market portfolio. When considering a sequential R&D process, we would need to model the evolution of the uncertainty about the value of the market portfolio. We could then use market information (for example, prices for futures and options on stock portfolios) to determine risk-neutral probabilities for the market uncertainty and use the integrated valuation procedure to determine optimal investment policies. As in the exploration and development model of Section 4.3, even though these market uncertainties will not affect the project cash flows until after the research is complete, as we “roll back the tree,” we may find that the market uncertainty impacts the values and policies even for early stage R&D projects. Just as we might terminate an oil exploration effort if oil prices were to fall, if we take the correlation between the value of a new drug and the market portfolio seriously, we might also terminate a pharmaceutical research project if the stock market were to drop unexpectedly.

5.3. Summary

In summary, we have shown how option pricing and decision analysis techniques can be integrated in evaluating oil properties and indicated how and why this integrated valuation methodology might be applied more broadly. The basic methodology is applicable whenever a firm (or individual) is contemplating investments with uncertain outcomes and there are market opportunities to hedge some of the risks associated with the investments. The methodology allows one to efficiently take into account information provided by the securities markets when valuing these kinds of projects and produces results that are consistent with those that would be produced given a detailed model of the securities markets.

APPENDIX

Proof of Proposition 1. Our proof proceeds in three steps. First we show that the discrete-time, finite-horizon approximate model, as defined in Equation (8), satisfies the following conditions:

- $v_{T,h}(t, x)$ is continuous and nondecreasing in x for each t .
- There exist thresholds $x_{T,h}^*(t)$ such that at time t it is optimal to abandon the property if and only if $x < x_{T,h}^*(t)$.

In this first step we explicitly construct and analyze the “binomial tree” approximation of the price and production rate processes. In the second step we take the limit as the horizon (T) recedes to infinity and show that, for fixed h , the values and policies approach stationary limits that satisfy conditions (a) and (b) and the limiting form of the recursion of equation (8). Finally, we take the limit as the discrete time step (h) approaches 0. The proof mimics standard proofs of dynamic programming, the unusual feature being the expected-effective-certainty-equivalent operator ($E^*[ECE[-]]$) used in the recursion (8).

Analysis of the Discrete-Time, Finite-Horizon Model. Let $x(t) \equiv p(t)q(t)$, where $p(t)$ is the price of oil and $q(t)$ is the production rate at time t . Following Cox et al. (1979), we approximate $p(t)$ and $q(t)$ by binomial processes whose mean and variance have been matched to underlying continuous time stochastic process. If $p(t)$ is the time- t price, then at time $t + h$,

$$p(t+h) = \begin{cases} p(t)(1 + \sigma_p \sqrt{h}) & \text{with probability } \frac{1}{2} \left(1 + \frac{\mu_p^*}{\sigma_p} \sqrt{h} \right), \\ p(t)(1 - \sigma_p \sqrt{h}) & \text{with probability } \frac{1}{2} \left(1 - \frac{\mu_p^*}{\sigma_p} \sqrt{h} \right). \end{cases}$$

The approximation for the production rate process $q(t)$ is similar. As $h \searrow 0$, these binomial processes converge to the continuous time limits specified by Equations (6a) and (6b).

Let $x \equiv x(t)$ and let $f(x)$ be any function of x . Substituting into Equation (8), and noting that $p(t)q(t) = x$, we have

$$\begin{aligned} M_{t,h}(f) &\equiv E^*[ECE_{t+h}[f(x(t+h))|p(t+h), q(t)]|p(t), q(t)] \\ &= -R_{t+h} \ln \left\{ \exp(-f(x(1 + \sigma_p \sqrt{h})(1 + \sigma_q \sqrt{h}))/R_{t+h}) \right. \\ &\quad \cdot \frac{1}{2} \left(1 + \frac{\mu_q}{\sigma_q} \sqrt{h} \right) + \exp(-f(x(1 + \sigma_p \sqrt{h})) \\ &\quad \cdot (1 - \sigma_q \sqrt{h})/R_{t+h}) \frac{1}{2} \left(1 - \frac{\mu_q}{\sigma_q} \sqrt{h} \right) \left. \right\} \\ &\quad \cdot \frac{1}{2} \left(1 + \frac{\mu_p^*}{\sigma_p} \sqrt{h} \right) + \\ &\quad - R_{t+h} \ln \left\{ \exp(-f(x(1 - \sigma_p \sqrt{h})(1 + \sigma_q \sqrt{h}))/R_{t+h}) \right. \\ &\quad \cdot \frac{1}{2} \left(1 + \frac{\mu_q}{\sigma_q} \sqrt{h} \right) + \exp(-f(x(1 - \sigma_p \sqrt{h})) \\ &\quad \cdot (1 - \sigma_q \sqrt{h})/R_{t+h}) \frac{1}{2} \left(1 - \frac{\mu_q}{\sigma_q} \sqrt{h} \right) \left. \right\} \\ &\quad \cdot \frac{1}{2} \left(1 - \frac{\mu_p^*}{\sigma_p} \sqrt{h} \right). \end{aligned}$$

It is now straightforward to check that if f is a continuous nondecreasing function of x , then so is $M_{t,h}(f)$. Define the mapping $L_{t,h}(f)$ via

$$L_{t,h}(f)(x) \equiv \max\{-c_a, c(x)h + e^{-rh}M_{t,h}(f)\}.$$

Because $-c_a$ and $-c(x)$ are continuous nondecreasing functions, it follows that $L_{t,h}$ maps continuous nondecreasing functions into continuous nondecreasing functions. Then, because $v_{T,h}(T, x)$ is continuous and nondecreasing in x , $v_{T,h}(t, x) = L_{t,h}(v_{T,h}(t+h, x))$ is also continuous and nondecreasing in x , thus establishing condition (a). Condition (b) then follows from the fact that $v_{T,h}(t, x)$ is nondecreasing in x .

Limit as $T \searrow \infty$. The next step is to let $v_h(x) \equiv \lim_{T \searrow \infty} v_{T,h}(0, x)$, and show that $v_h(x)$ is a continuous, nondecreasing function of x satisfying the recursion

$$\begin{aligned} v_h(x) &= \max\{-c_a, c(x)h + e^{-rh} E^* \\ &\quad [ECE[\\ &\quad v_h(x(t+h))|p(t+h), q(t)]|p(t), q(t)]\}. \end{aligned}$$

Define the mapping $L_h(f)$ via

$$\begin{aligned} L_h(f)(x) &\equiv \max\{-c_a, -c(x)h + e^{-rh} E^*[ECE[\\ &\quad f(x(t+h))|p(t+h), q(t)]|p(t), q(t)]\}. \end{aligned}$$

By the “ Δ -property” of the exponential utility function used to calculate effective certainty equivalents, for any constant Δ , $ECE[f(x(t+h)) + \Delta|p(t+h), q(t)] = ECE[f(x(t+h))|p(t+h), q(t)] + \Delta$. Substituting into the

definition of L_h yields $L_h(f + \Delta) = L_h(f) + e^{-rh} \Delta$; so L_h is a contraction mapping. It then follows (see Stokey and Lucas 1989, p. 270–272 or Lippman 1975) that there is a unique continuous, nondecreasing function $v_h(x)$ satisfying $v_h(x) = L_h(v_h(x))$, and, furthermore, $v_h(x) = \lim_{T \rightarrow \infty} v_{T,h}(0, x)$. Because L is monotone ($L_h(f) \geq L_h(g)$ for $f \geq g$), it follows that $v_{T,h}(0, x)$ is nondecreasing in T for each x . This then implies that $x_{T,h}^*(0)$ is nonincreasing in T and converges to x_h^* , the abandonment threshold for the infinite horizon case.

Limit as $h \rightarrow 0$. Let $v(x) \equiv \lim_{h \rightarrow 0} v_h(x)$. Because each $v_h(x)$ is nondecreasing in x , it follows that $v(x)$ is also nondecreasing. Furthermore, because each $v_h(x)$ is continuous and the slope of each $v_h(x)$ is bounded above by $\gamma/(r - \mu^*)$, the slope of the risk-neutral solution without abandonment, it follows that $v_h(x)$ converges to a continuous, nondecreasing function $v(x)$. Thus there is a threshold x^* such that it is optimal to abandon if $x < x^*$ and to continue production otherwise. At this threshold, we have $v(x^*) = -c_a$, or if $x^* \leq 0$, $v(0) = -c_a/r$. The smooth pasting condition, $v'(x^*) = 0$ (for $x^* > 0$), then follows as in Dixit and Pindyck (1994, p. 130–132).

To derive Equation (9), multiply both sides of the recursion

$$v_h(x) = \max\{-c_a, c(x)h + e^{-rh} E^*[ECE[v_h(x(t+h))|p(t+h), q(t)]|p(t), q(t)]\},$$

by e^{rh} and rearrange terms using the Δ -property ($v_h(x) = v_h(x(t))$ is a constant given the current prices and production rates) to yield

$$(e^{rh} - 1)v_h(x) = \max\{-e^{rh}c_a - v_h(x), e^{rh}c(x)h + E^*[ECE[v_h(x(t+h)) - v_h(x)|p(t+h), q(t)]|p(t), q(t)]\}.$$

Divide both sides by h and take the limit as limit as $h \rightarrow 0$. If $x > x^*$, then when divided by h the first term in the maximization goes to $-\infty$ as h goes to 0, and the second term goes to $c(x) + (1/dt) E^*[ECE [dv(x)]]$. If $x < x^*$, then when divided by h , the first term goes to $-rc_a$. This completes the proof of Proposition 1. \square

Derivation of Equation (10). Taking $t = 0$ and writing x for $x(0)$ and substituting Pratt's (1964) approximation for the effective certainty equivalent into Equation (8), we have

$$v(x) = hc(x) + e^{-rh} \left(E^*[v(x(h))|x(0)] - \frac{1}{2R} E^*[\text{Var}[v(x(h))|p(h), q(0)]] \right) + o(h). \tag{A1}$$

Following Karlin and Taylor (1981, p. 203–204) (essentially applying Ito's Lemma), we have

$$E^*[v(x(h))|x(0)] = v(x) + h\mu_x^*xv'(x) + h\frac{\sigma_x^2}{2}x^2v''(x) + o(h),$$

and similarly,

$$\text{Var}[v(x(h))|p(h), q(0)] = E[v^2(x(h))|p(h), q(0)] - (E[v(x(h))|p(h), q(0)])^2 = (v^2(x) + 2h\mu_qxv(x)v'(x) + h\sigma_q^2x^2(v(x)v''(x) + v'(x)^2)) - (v^2(x) + 2h\mu_qxv(x)v'(x) + h\sigma_q^2x^2v(x)v''(x)) + o(h) = h\sigma_q^2x^2v'(x)^2 + o(h).$$

Substituting back into Equation (A1), this becomes

$$v(x) = hc(x) + e^{-rh} \left(v(x) + h\mu_x^*xv'(x) + \frac{h}{2} \sigma_x^2x^2v''(x) - \frac{h}{2R} \sigma_q^2x^2v'(x)^2 \right) + o(h).$$

Rearranging, dividing through by h , and taking the limit as h goes to 0, we get Equation (10) in the text.///

ENDNOTES

1. This may be viewed as a consequence of the smooth pasting condition of Proposition 1, which says that the value function is flat at the threshold corresponding to the optimal policy. Since the myopic value function lies between the optimal value function and $-c_a$, the differences between the two value functions will be small.
2. The optimal policy calls for drilling 15 wells because the optimal drilling threshold given 0 wells producing ($x_d(0)$) exceeds the drilling thresholds given 1–14 wells producing, but is less than the threshold given 15 wells producing. Also note that Figure 3 shows only every fifth change in optimal policy; there are four changes in policy between each of the crosses shown in the figure.

ACKNOWLEDGMENT

We are grateful for helpful comments from Rick Chamberlain, Bernard Dumas, Gardner Walkup, Bob Winkler and two anonymous referees. The authors gratefully acknowledge the support of the National Science Foundation under grant no. SBR-9511364.

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